9210-100
Level 6 Graduate Diploma in Engineering
Engineering mathematics

Sample Paper

You should have the following for this examination
- one answer book
- drawing instruments
- non-programmable calculator
- statistical table

General instructions
- This examination paper is of **three hours** duration.
- This examination paper contains **nine** questions in **three** sections.
- Answer **five** questions selecting at least one from each section.
- All questions carry equal marks. The maximum marks for each section within a question are given against that section.
- An electronic, non-programmable calculator may be used, but the candidate **must** show clearly the steps prior to obtaining final numerical values.
- Drawing should be clear, in good proportion and in pencil. Do **not** use red ink.
- All pressures are absolute unless otherwise stated.

Section A

1 a) i) If $y(x) = \int_0^x f(t)dt$, show that

$$\frac{dy}{dx} = f(x)$$ (4 marks)

[Consider $\lim_{\triangle x \to o} \frac{1}{\triangle x}\{\int_0^{x+\triangle x} f(t)dt - \int_0^x f(t)dt\}$]

ii) If $u(x) = e^{x^2} \int_0^x e^{-t^2}\, dt$, show that u(x) satisfies the differential equation (4 marks)

$$\frac{du}{dx} = 1 + 2ux, x(0) = 0.$$

b) Potential V(x, y) at any point (x, y) of a charged plate is given by

$V(x, y) = xy^2 - x^2y - 4x, \quad x, y > 0.$

i) Determine the coordinates of the point or points at which V is stationary. (4 marks)

ii) Hence identify the nature of the stationary points. (4 marks)

iii) Use Lagrange multipliers method to determine stationary points of V on the line x + y = 3. (4 marks)

2 a) State the condition for a vector in a field to be conservative. (2 marks)

b) i) If $\mathbf{F} = 2(xy + z)\mathbf{I} + x^2\,\mathbf{j} + 2x\,\mathbf{K}$, determine div **F** and curl **F** (4 marks)

ii) Find ǿ such that **F** = grad ǿ, where **F** is as given above. (4 marks)

iii) Evaluate $\iint_S (\mathbf{F}.\mathbf{n})ds$, where **F** is as given above with S the surface of $V = [\,0 \le x \le 1,\, 0 \le y \le 1,\, 0 \le z \le 1]$ (6 marks)

iv) Write down the value of the integral

$\oint_C \mathbf{F}.d\mathbf{r}$

where C is the closed path made by V above on the x-y plane, with **F** given as above and $\mathbf{r} = x\,\mathbf{I} + y\,\mathbf{j} + z\,\mathbf{K}$, and integration is conducted counter-clockwise. (4 marks)

3 a) i) State Cauchy- Riemann equations. (2 marks)

ii) If w = u(x, y) + iv(x, y), is an analytic function with $u = e^{-x}\sin y$, determine v such that v = 0, when x = 0. (6 marks)

iii) Hence show that $w(z) = ie^{-z}$, where z = x + iy. (2 marks)

b) Taking $C : |z - a| \le a/2$, and integration taken in the counter-clockwise sense evaluate the integrals

i) $$\oint_C \frac{e^{-z}}{(z^2 - a^2)}\, dz$$ (5 marks)

ii) $$\oint_C \frac{e^{-z}}{(z^2 - a^2)^2}\, dz$$ (5 marks)

4 a) i) If a function f(x) = cos x, $-1 \le x \le 1$, with f(x + 2) = f(x), is expressed using Fourier half range sine series as

$f(x) = \sum_{n=1}^{\infty} b_n \sin n\pi x$

state the expression for b_n. (2 marks)

ii) Derive the half range Fourier sine series for the function f(x) is given as f(x) = cos x, $0 < x < 1$, (6 marks)

b) The function u = u(x, t) satisfies the equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \; 0 \le x \le 1,\; t \ge 0,$$

U(0, t) = 0, u(1, t) = 0, $t \ge 0$.

i) Show by using the variables separable method that u(x, t) can be expressed as $u(x, t) = \sum_{n=1}^{\infty} b_n e^{-n^2\pi^2 t} \sin n\pi x$, and b_n are arbitrary constants. (6 marks)

ii) Further, if u(x, 0) = f(x), with f(x) given in a)ii) above, determine the first three terms of the solution for u(x, t). (6 marks)

5 a) i) If Laplace Transform of f (t) is given as

L{ f(t) } = F(s),

determine $L\{\frac{d}{dt}\}$, given f(0). (4 marks)

ii) Also by using the result $\frac{d}{dt}\left[\int_0^t f(\tau)d\tau\right] = f(t)$, show that

$$L\{\int_0^t f(\tau)d\tau\} = \frac{F(s)}{s}$$ (4 marks)

iii) The current i(t) in the circuit shown in Figure Q5 satisfies the equation

$$R.i(t) + \frac{1}{C}\int_0^t i(\tau)d\tau + L\frac{di}{dt} = v(t)$$

where L, R, and C, are inductance, resistance and capacitance respectively and v the applied voltage in the circuit. (4 marks)

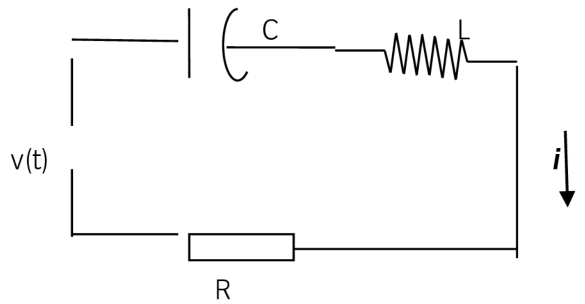

Figure Q5

By taking Laplace Transform of terms of equation for i(t) in a)iii), and with I(s) = L{f(t)}, show that the equation reduces to

$$(Ls^2 + Rs + \frac{1}{C})I = L\{\frac{dV}{dt}\} + sLI'(0) + (sL + R)i(0)$$ (4 marks)

[Formula $L\{\frac{f''(t)}{dt^2}\} = s^2F(s) - sf'(0) - f(0)$, F(s) = L{f(t)} may be used.]

iv) Taking C = 1/4 , R = 4, L = 1 , and v(t) = v(t) = t γ(0), i(0) = 0 , i'(0) = 0 where γ(0) is the unit impulse function at t = 0. Determine the expression for i(t) for t > 0. [Take L{γ(0)} = 1 and γ'(0) = 0.] (8 marks)

Section B

6 a) i) A function u(t) satisfies the differential equation

$$\frac{du}{dt} = f(t), \text{ with } u = u_0, \text{ at } t = t_0, \text{ where } f(t, u) \text{ is a given function.}$$

Write down the 2nd order Taylor series solution for $u(t_0 + h)$. (4 marks)

ii) State the error term of the above expansion. (2 marks)

iii) Second order Runge-Kutta (RK2) to obtain the solution at $t = t_0 + h$, for given small h,
Is given as

$$k_1 = hf(t_0, u_0)\,), \; k_2 = hf(t_0 + h, u_0 + k_1),$$

$$u_1 = u_0 + \frac{1}{2}(k_1 + k_2)$$

Write down the equivalent RK2 formuli for solutions at $t = t_0 + h$, for the system of equations

$$\frac{du}{dt} = f(t, u, v),$$

$$\frac{dv}{dt} = g(t, u, v), \text{ with } u = u_0, v = v_0, \text{ at } t = t_0.$$ (4 marks)

b) i) A meteorological equipment fixed with a mild parachute mechanism is dropped from a height and the distance travelled by the instrument y(t) in time t satisfies the equation

$$\frac{d^2y}{dt^2} = g - \alpha\frac{dy}{dt}, \; y(0) = 0, \; y'(0) = 0$$

By taking y' = v, and y" = v', write down the given equation in terms of a pair of first order differential equations. (4 marks)

ii) Taking g = 10 , and $\alpha = 0.1$ in appropriate units, determine by use of the RK2 method y and v, at t = 10, 20 units of time. (6 marks)

7 a) i) For u = u(x, t), write down the explicit scheme for the numerical solution of the equation

$$\frac{du}{dt} = \frac{\partial^2 u}{\partial x^2}, 0 \le x \le 1, t > 0,$$

u(0, t) = u(1, t) = 0, t ≥ 0,

u(x, 0) = g(x), given g(x), $0 \le x \le 1$. (4 marks)

ii) Explain briefly the advantage and disadvantage of using explicit scheme and the implicit scheme for the solution of the above problem. (4 marks)

b) i) The temperature u(x, t) of a rod initially heated and allowed to cool with two ends kept at zero temperature. The problem is scaled and modeled for the temperature to satisfy the differential equation and boundary conditions given in a)i) above, with

$$g(x) = \begin{cases} (\frac{1}{6}) \times x, & 0 \le x \le 0.6 \\ (\frac{1}{6}) \times (1 - x), & 0.6 \le x \le 1. \end{cases}$$

Obtain the numerical solution for u(x, t), x = x_i = 0.2 x *i*, = 0, 1, 2, ..5 and t = t_j = 0.01 xj, j = 1, 2, 3. (8 marks)

ii) Explain briefly the nature of solution that can be expected for large values of t. (2 marks)

iii) Further explain the nature of solution for t = tj = 0.02 x j, j = 1, 2, …. with the other conditions remaining the same. (2 marks)

Section C

8 a) i) Given that Poisson's distribution is given as the probability of a number r occurrences of an event is given by

$$P(r) = \frac{e^{-\lambda}\lambda^r}{r!}, \ r = 0, 1, 2, \ldots.$$

state the (1) mean, and (2) standard deviation, of the distribution. (2 marks)

ii) A factory is subjected to an average of one calamity every year. Determine the probability that for any year the number of calamities will be (1) 0, (2) 1, (3) more than 1. (4 marks)

iii) If the factory undergoes a loss of $1000 and $2000, for 1 calamity and more than 1 calamity respectively, estimate the average loss the factory will incur every year. (4 marks)

b) A factory manufactures a type of bolt with mean diameter of 2.0 cm and standard deviation of 0.2 cm. Assume that diameters are normally distributed.

i) Samples of 16 bolts are taken at random from the production line every two hours and mean value of the diameters of the bolts is calculated for the purpose of maintaining quality of products. Design a control chart to examine whether the mean diameter of the samples will fall between control limits with 95 % probability. (4 marks)

ii) Following modifications conducted on the production process it is found that bolts have mean diameter 2.01 cm and standard deviation 0.18 cm. Conduct a hypothesis test at 95 % confidence limit to examine whether there is a change of the diameters due to the modifications. (6 marks)

9 a) A line $y = a + bx$, is to be fitted to n points with coordinates (x_i, y_i)

Taking the value of $b = \frac{\{\sum xy - n\bar{x}\,\bar{y}\}}{\{\sum x^2 - n\bar{x}^2\}}$, show how to calculate the value of the parameter a. (4 marks)

b) In a certain hospital the numbers of outdoor patients receiving general treatment and clinical treatment per day for five years are given in ' 000s in the Table Q9.

Year (t)	1	2	3	4	5
General Treatment (x)	1.4	1.7	1.7	1.8	1.9
Clinical Visits (y)	2.9	2.7	2.6	3.0	2.9

Table Q9

i) Given that $\sum x = 8.5$, $\sum y = 14.1$, $\sum x^2 = 14.59$, $\sum y^2 = 39.87$, $\sum x\,y = 23.98$, determine the correlation coefficient r, between numbers coming for general treatment and numbers for clinical visits.
$[r = (\sum xy - n\bar{x}\,\bar{y})/((\sum x^2 - n\bar{x}^2) * (\sum y^2 - n\bar{y}^2))^{1/2}]$ (4 marks)

ii) Also calculate parameters a and b to fit a straight line for values x and y. (6 marks)

iii) By taking the base year as the year 3, ($T = t - 3$) fit a straight line for x as $x = a' + b'\,T$. (4 marks)

iv) Hence estimate x for year T = 10 and determine y for same year T = 10. (2 marks)