Introduction

This scheme of work is intended to assist the tutor in delivering this unit to engineering students. It includes a considerable number of worked examples as an aide memoire for the tutor, and a number of suggestions for practice problems. However, because the primary aim is to teach the mathematical methods, there is often little engineering context. It is extremely important that, once each mathematical process has been mastered, students should be introduced to contextual engineering problems so that they can have practice in

- Selecting the correct method and
- Working towards a recognisable engineering solution to an engineering problem. This is important in order to maintain students' motivation and focus

Suggested teaching times

Since each student group is unique, the suggested teaching hours can only be a guide as to how the allotted 60 GLH is divided up. If there is time left on a session then students should fill that time doing realistic problems until they become second nature. Students should appreciate that in engineering, mathematics is an essential and very powerful tool. It is not done for its own sake, but as a means of solving complex engineering problems.

Lesson 1: Working with algebraic functions **Suggested Teaching Time:** 8 hours

Learning Outcome: 1. Be able to use algebraic methods to analyse and solve engineering problems

Topic	Suggested Teaching	Suggested Resources
AC 1.1 Evaluate **basic algebraic functions** **AC 1.2** Solve engineering problems that are described by **algebraic equations** and **exponential or logarithmic functions**	**Session 1** (**2 hrs**)**: Functions** • Explain what is meant by a function and a composite function. • Solve problems involving composite functions. g·f(x) is a composite function and when calculating this type, f is performed first and the result is substituted into g **Explain what is meant by an inverse function.** **Solve problems involving inverse functions.** In general the inverse of a function is the reverse of the operation carried out to find $f(x)$ and is written as $\boldsymbol{f^{-1}(x)}$ Given $\boldsymbol{f(x) = 2x + 1}$ find $\boldsymbol{f^{-1}(x)}$ Let $\boldsymbol{y = 2x + 1}$ change $\boldsymbol{f(x)}$ as $\boldsymbol{y}$ and $\boldsymbol{f^{-1}(y) = x}$ $\boldsymbol{y = 2x + 1}$ $\boldsymbol{x = \frac{y-1}{2}}$ $\therefore \boldsymbol{f^{-1}(x) = \frac{x-1}{4}}$	**Book:** Bird. J. O., *Higher Engineering Mathematics* 7th edition (Routledge 2014) ISBN-13: 978-0415662826 **Websites:** http://www.mathcentre.ac.uk/links http://mathworld.wolfram.com/ http://www.mathcentre.ac.uk/ Good free UK-based learning resource for L4/5 maths

Lesson 1: Working with algebraic functions **Suggested Teaching Time:** 8 hours

Learning Outcome: 1. Be able to use algebraic methods to analyse and solve engineering problems

Topic	Suggested Teaching	Suggested Resources
	Session 2 (2 hrs): Indices Revise the laws of indices. Solve equations involving indices. **e.g.** $125^{(2x+1)} \cdot 25^{2x} \cdot 625^{(3x+1)} = 3125$	
	Session 3 (2 hrs): Partial fractions Rules and methods for partial fractions. Solve problems in partial fractions: Three basic types of denominators • Linear factors: $\frac{3}{(x+5)(x-2)} = \frac{A}{(x+5)} + \frac{B}{(x-2)}$ • Quadratic factor that does not factorise: $\frac{(2x+3)}{(x-4)(x+7)^2} = \frac{A}{(x+3)} + \frac{Bx+C}{(x^2+5)}$	

Lesson 1: Working with algebraic functions **Suggested Teaching Time:** 8 hours

Learning Outcome: 1. Be able to use algebraic methods to analyse and solve engineering problems

Topic	Suggested Teaching	Suggested Resources
	Repeated root $$\frac{(2x+3)}{(x-4)(x+7)^2} = \frac{A}{(x-4)} + \frac{B}{(x+7)} + \frac{C}{(x+7)^2}$$ Explain to students the use of partial fractions to enable some integration to be carried out.	
	Session 4 (2 hrs): Logarithms State the laws of logarithms. Apply the laws to solve problems. Express in terms of log a, log b and log c $\log(a^3bc^4)$ and $\log\frac{\sqrt{10b}}{c^3}$ Solve for x: $(5.932)^{2x-4} = (9.875)x$ The charge q on a capacitor is given by: $q = 3.16 \times 2.72^{-4.32t}$ Calculate: i) t when q = 1.7 ii) q when t = 0.28	

Lesson 2: Analysis using trigonometry

Suggested Teaching Time: 5 hours

Learning Outcome: 2. Be able to solve engineering problems that require the use of trigonometric methods of analysis

Topic	Suggested Teaching	Suggested Resources
AC 2.1 Evaluate basic trigonometric functions **AC 2.2** Evaluate **trigonometric identities** to solve problems involving trigonometric equations.	**Session 1 (5 hrs): Trigonometry** Revision of sine and cosine rules, trigonometric graphs and sec, cosec and cotan. State and apply trigonometric identities, addition and double-angle formulae: Addition formulæ: $\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$ $\cos(A + B) = \cos A \cos B \pm \sin A \sin B$ $\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$ $\tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$ Double-angle formulæ: $\sin A = 2 \sin A \cos A$ $\cos 2A = \cos^2 A \sin^2 A$ $\tan 2A = \frac{2 \tan A}{1 - \tan^2 A}$	**Book:** Bird. J. O., *Higher Engineering Mathematics* 7th edition (Routledge 2014) ISBN-13: 978-0415662826 **Websites:** http://www.mathcentre.ac.uk/links http://mathworld.wolfram.com http://www.mathcentre.ac.uk/

Lesson 2: Analysis using trigonometry

Suggested Teaching Time: 5 hours

Learning Outcome: 2. Be able to solve engineering problems that require the use of trigonometric methods of analysis

Topic	Suggested Teaching	Suggested Resources
	Trigonometric identities: Identities $\sin x = \tan x \,.\cos x$ $\cos^{-1} x + \sin^2 x = 1$ $1 + \tan^2 x = \sec^2 x$ $1 + \cot^2 x = \mathrm{cosec}^2\, x$ Solve trigonometric equations within given ranges both in degrees and radians. Determine exact values (surd form) of 15°, 30˚, 45˚ etc. Conversion of $a \sin \omega t + b \cos \omega t$ into $R \sin(\omega t + \alpha)$ Solve problems in three-dimensional trigonometry with an engineering context; an example would be finding the resultant of three forces acting on a cutting tool.	

Lesson 3: Differential Calculus **Suggested Teaching Time:** 8 hours

Learning Outcome: 3. Be able to use methods of differential and integral calculus to solve engineering problems

Topic	Suggested Teaching	Suggested Resources
AC 3.1 Evaluate first and higher order derivatives of a function involving algebraic and/or trigonometric expressions **AC 3.2** Use differential calculus to obtain solutions for engineering applications of algebraic and trigonometric equations	**Session 1 (2 hrs): Differentiation** Revise rules and methods of differentiation of simple algebraic functions. Solve equations for maximum or minimum values Apply the rules to find higher derivatives $\frac{d^2y}{dx^2}$, $\frac{d^3y}{dx^3}$ **etc.**	**Book:** Bird. J. O., *Higher Engineering Mathematics* 7th edition (Routledge 2014) ISBN-13: 978-0415662826 **Websites:** http://www.mathcentre.ac.uk/links http://mathworld.wolfram.com/ http://www.mathcentre.ac.uk/
	Session 2 (2 hrs): Differentiation of trigonometric functions. Differentiate: $y = a \sin x$ $y = \tan a\theta$ $y = \sin a\theta$ $y = \tan \frac{a\theta}{b}$ $y = \operatorname{cosec} \frac{\theta}{a}$ $y = \sin (a\theta + x)$	

Lesson 3: Differential Calculus **Suggested Teaching Time:** 8 hours

Learning Outcome: 3. Be able to use methods of differential and integral calculus to solve engineering problems

Topic	Suggested Teaching	Suggested Resources
	Session 3 (2 hrs): Differentiation of logarithmic and exponential functions Explain differential properties of logarithmic and exponential functions. Differentiate: $y = \ln x$ $y = \ln 5x$ $y = 4\ln 3x$ $y = \ln ae^{-3x}$ Solve problems involving for example: Discharge of a capacitor: $q = Qe^{\frac{-t}{RC}}$ Tension in belts: $T_1 = T_2e^{\mu\theta}$ Growth of current in a capacitive circuit: $i = i(1 - e^{\frac{-t}{RC}})$ Differentiate: $y = \ln(6x^3 - 5)$ $y = \ln(\sin 3x)$ $y = \ln\frac{x+4}{x-3}$ use $\log\frac{a}{b} = \log a - \log b$ in the third example	

Lesson 3: Differential Calculus **Suggested Teaching Time:** 8 hours

Learning Outcome: 3. Be able to use methods of differential and integral calculus to solve engineering problems

Topic	Suggested Teaching	Suggested Resources
	Session 4 (2 hrs): Differentiation of a function of a function. The chain rule $$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$ Differentiate $y = (4x^3 - 5)^4$ Let $(4x^3 - 5) = u$ then $y = u^4$ $$\frac{du}{dx} = 12x^2 \text{ and } \frac{dy}{du} = 4u^3$$ $then$ $$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = 4u^3 \cdot 12x^2$$ $$= 48x^2(4x^3 - 5)^3$$ An easier method is to differentiate the bracket, treating it as x^n then differentiate the function inside the bracket. To obtain $\frac{dy}{dx}$ multiply the two results together. Check back to the answer above. Differentiate different types e.g. $$y = \sqrt{4x^3 + 5x - 4}$$ $$y = \frac{3}{(4t^3 - 7)^5}$$	

Lesson 3: Differential Calculus **Suggested Teaching Time:** 8 hours

Learning Outcome: 3. Be able to use methods of differential and integral calculus to solve engineering problems

Topic	Suggested Teaching	Suggested Resources
	Session 5 (2 hrs): Differentiation of a product When $y = uv$ and u and v are functions of x then $$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$ In words it can be remembered as the first times the derivative of the second plus the second times the derivative of the first. Solve problems involving logarithmic, trigonometric and exponential terms for example: a) $$y = 3x^3 sin2x$$ $$\frac{dy}{dx} = (3x^3)(2cos2x) + (sin2x)(9x^2)$$ $$= 6x^3 cos2x + 9x^2 sin2x$$ $$= 3x^2(2xcos2x + 3sin2x)$$ b) Given that $$V = 2e^{3t} sin2t,$$ evaluate $\frac{dv}{dt}$ when $t = 0.5$ $$\frac{dv}{dt} = 2e^{3t} \cdot 2cos2t + sin2t \cdot 6e^{3t}$$	

Lesson 3: Differential Calculus **Suggested Teaching Time:** 8 hours

Learning Outcome: 3. Be able to use methods of differential and integral calculus to solve engineering problems

Topic	Suggested Teaching	Suggested Resources
	Substitute for t: $$2e^{1.5}.2cos1 + sin1 \cdot 6e^{1.5}$$ **= 32.31** (note that sin and cos are in radians)	
	Session 6 (**2 hrs**): **Differentiation of a quotient** When: $y = \frac{u}{v}$ and u and v are functions of x then $$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$ Solve problems involving logarithmic, trigonometric and exponential terms for example: $a)$ $$y = \frac{3sin4x}{4x^3}$$ $$\frac{dy}{dx} = \frac{4x^3 \cdot 12cos4x - 3sin4x \cdot 12x^2}{16x^6}$$ $$= \frac{48x^3cos4x - 36x^2sin4x}{16x^6}$$ $$= \frac{3(4xcos4x - 3sin4x)}{4x^4}$$	

Lesson 3: Differential Calculus **Suggested Teaching Time:** 8 hours

Learning Outcome: 3. Be able to use methods of differential and integral calculus to solve engineering problems

Topic	Suggested Teaching	Suggested Resources
	b) $$y = \frac{2xe^{4x}}{sinx}$$ $$= \frac{sinx(2x \cdot 4e^{4x} + 2e^{4x}) - 2xe^{4x}cosx}{\sin^2 x}$$ $$= \frac{8xe^{4x} \cdot sinx + 2e^{4x}sinx - xe^{4x}\,cosx}{sin^2x}$$ $$= \frac{2e^{4x}(4xsinx + sinx - xcosx)}{sin^2x}$$ $$= \frac{2e^{4x}}{sin^2x}[sinx(4x+1) - xcosx)]$$	
	Session 7 (**2 hrs**): **Differentiating implicit functions** When equations involving say x and y this is called an implicit function e.g. $$y^4 + 3x^2 = y^3 - x$$ By using the **function of a function rule** it is possible to differentiate an implicit function. Differentiating an implicit function can be summarised as $$\frac{d}{dxf(y)} = d/dyf(y) \cdot \frac{dy}{dx}$$	

Lesson 3: Differential Calculus **Suggested Teaching Time:** 8 hours

Learning Outcome: 3. Be able to use methods of differential and integral calculus to solve engineering problems

Topic	Suggested Teaching	Suggested Resources
	Solving typical problems where the answers should be in their simplest form: Find $\frac{dy}{dx}$ **in terms of** x **and** y a) $y^4 + 8x = x^3$ $4y^3\frac{dy}{dx} + 8 = 3x^2$ $\frac{dy}{dx} = \frac{3x^2 - 8}{4y^3}$ b) $x^3 + 3xy - y^2 = 6$ using the product rule: $3x^2 + 3x\frac{dy}{dx}(\times 1) + y(\times 3) - 2y\frac{dy}{dx} = 0$ $\frac{dy}{dx}(3x - 2y) = 3(y - x^2)$ $\frac{dy}{dx} = \frac{3(y - x^2)}{(3x - 2y)}$	

Lesson 4: Integral Calculus

Suggested Teaching Time: 16 hours

Learning Outcome: 3. Be able to use methods of differential and integral calculus to solve engineering problems

Topic	Suggested Teaching	Suggested Resources
AC 3.3 Use methods of integration to determine indefinite and definite integrals of algebraic and trigonometric functions **AC 3.4** Use integral calculus to obtain solutions for engineering applications of algebraic and trigonometric equations **AC 3.5** Use **integration** to solve engineering applications of differential equations in which the variables are separable.	**Session 1 (2 hrs): Integration. Revision of basic concepts, methods and rules of integration.** Revision of: $\int \frac{1}{x^4} + \frac{1}{x^2} + x^4 + x^2 dx$ $\int_0^1 5cos3xdx$ $\int_{-1}^{2} \frac{2}{3e^{2x}} dx$ $\int \frac{x^4+2}{x^2} dx$ Integration by substitution and by parts $\int 2x(x^3-4)dx$ Practice problems.	**Book:** Bird. J. O., *Higher Engineering Mathematics* 7th edition (Routledge 2014) ISBN-13: 978-0415662826 **Websites:** http://www.mathcentre.ac.uk/links http://mathworld.wolfram.com/ http://www.mathcentre.ac.uk/

Lesson 4: Integral Calculus **Suggested Teaching Time:** 16 hours

Learning Outcome: 3. Be able to use methods of differential and integral calculus to solve engineering problems

Topic	Suggested Teaching	Suggested Resources
	Session 2 (**3 hrs): Area under a curve.** Sketch graphs of functions to be able to answer questions about areas under the graph between given x or y values. Sketch the graph of: $$y = x^3 + 2x^2 - 5x - 6$$ Show how integration can be used to calculate areas bounded by a curve and an axis. Use definite integration to find the area between an axis and a curve lying above the x-axis. Use definite integration to find the area between an axis and a curve lying below the x axis Determine the points of intersection of two curves and the area between these points. **Note that:** Confusion can occur when calculating areas that are above and below the axis. The areas should be calculated separately and then added together.	

Lesson 4: Integral Calculus **Suggested Teaching Time:** 16 hours

Learning Outcome: 3. Be able to use methods of differential and integral calculus to solve engineering problems

Topic	Suggested Teaching	Suggested Resources
	Find the area between $y = x$ and $y = x^2$ from $x = 0$ and $x = 1$ $$A = \int_0^1 [x - x^2]dx$$ $$= \left[\frac{1}{2}x^2 - \frac{1}{3}x^3\right]_0^1$$ $$= \frac{1}{6}\ sq\ units$$ Determine the points of intersection of the following two curves and the area between these points: $y^2 = 3x$ and $x^2 = 3y$ $$y = 3x^{\frac{1}{2}} \text{ and } y = \frac{x^2}{3} \therefore 3(3x)^{\frac{1}{2}} = x^2$$ Square both sides: $9 \cdot 3x = x^4 \therefore x^3 = 27\ \ so\ x = 3\ or\ 0$ $$A = \int_0^3 3x^{\frac{1}{2}} - \frac{x^2}{3}dx$$ $$A = \left[2x^{\frac{3}{2}} - \frac{x^3}{9}\right]_0^3$$ $$= 10.4\ square\ units$$	

Lesson 4: Integral Calculus

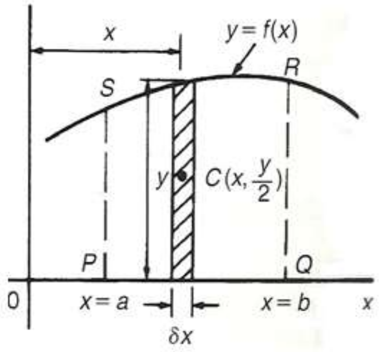
Suggested Teaching Time: 16 hours

Learning Outcome: 3. Be able to use methods of differential and integral calculus to solve engineering problems

Topic	Suggested Teaching	Suggested Resources
	Session 3 (**2 hrs**)**: Volumes of rotation** Some practical examples should be used e.g.: A bucket has top and bottom radii of 200mm and 100mm respectively and a height of 200mm. a) Show, when the sides slope uniformly, that it may be considered as being formed by the revolution of the line $y = \frac{x}{2} + 100$ about the $\mathbf{x}$ axis from $\mathbf{x} = 0$ to $\mathbf{x} = 200$ b) Find the capacity of the bucket in litres a) $$y - mx + y = \frac{100}{200x} + 100$$ $$y = \frac{x}{2} + 100$$ b) $$V = \pi \int_0^{200} y^2 dx$$ $$\pi \left[\frac{x^3}{3} + 50x^2 + 10000x\right]_0^{200} \qquad = 14.7\ litres$$	

Lesson 4: Integral Calculus **Suggested Teaching Time:** 16 hours

Learning Outcome: 3. Be able to use methods of differential and integral calculus to solve engineering problems

Topic	Suggested Teaching	Suggested Resources
	Session 4 (**2 hrs**): **Centre of mass of a lamina** Show that the centre of mass of a lamina is  $$x = \frac{\int_a^b xydx}{\int_a^b ydx}$$	
	Session 5 (**2 hrs**): **Mean value under a curve.** $$\bar{y} = \frac{\frac{1}{2}\int_a^b y^2dx}{\int_a^b ydx}$$	

Lesson 4: Integral Calculus

Suggested Teaching Time: 16 hours

Learning Outcome: 3. Be able to use methods of differential and integral calculus to solve engineering problems

Topic	Suggested Teaching	Suggested Resources
	Show that the mean value under a curve $$\bar{y} = \frac{area\ under\ curve}{length\ of\ the\ base}$$ If the area of the curve is found by integration then $$\bar{y} = \frac{\int_a^b ydx}{b-a}\quad or$$ $$\bar{y} = \frac{1}{b-a}\int_a^b f(x)dx$$ Solve problems e.g.: Determine the co-ordinates of the centre of area of the curve $y = 5x^2$ between $x = 1$ and $x = 4$ Example 1 Determine, by integration, the mean value of $y = 3x^3$ between $x = 1$ and $x = 3$ $$\bar{y} = \frac{1}{3-1}\int_1^3 ydx$$ $$\bar{y} = \frac{1}{2}\int_1^3 3x^2dx$$ $$= \frac{1}{2}\left[\frac{3x^4}{4}\right]_1^3 = 30$$	

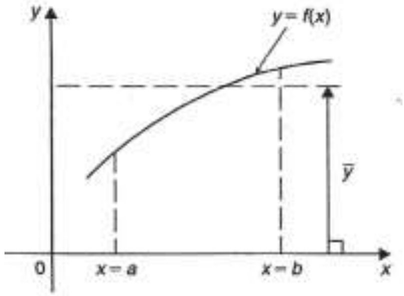
Lesson 4: Integral Calculus **Suggested Teaching Time:** 16 hours

Learning Outcome: 3. Be able to use methods of differential and integral calculus to solve engineering problems

Topic	Suggested Teaching	Suggested Resources
	Example 2 A sinusoidal voltage $\boldsymbol{v = 100 sin\omega t}$. Using integration determine the mean voltage over a half cycle. Note: When finding the mean value of a periodic function such as a sine wave the mean value is taken over a half cycle because the mean value over a complete cycle is zero. $\bar{v} = \frac{1}{\pi - 0}\int_0^{\pi} v\,d\omega t$ $= \frac{1}{\pi}\int_0^{\pi}(100 sin\,\omega t)\,d\omega t$ $= \frac{100}{\pi}\left[-cos\omega t\right]_0^{\pi}$ Practice problems using the following examples as a guide: a) Determine the mean value of $y = sin2\theta$ from $\theta = 0\ to\ \theta = \frac{\pi}{4}$ b) A sinusoidal voltage has a peak value of **350v**, calculate its mean value.	

Lesson 4: Integral Calculus **Suggested Teaching Time:** 16 hours

Learning Outcome: 3. Be able to use methods of differential and integral calculus to solve engineering problems

Topic	Suggested Teaching	Suggested Resources
	Session 6 (**2 hrs): root mean square (RMS) values** The RMS value of an alternating current is defined as that current which will give the same heating effect as the equivalent direct current.  $$r.m.s = \sqrt{\frac{1}{b-a}\int_a^b y^2\,dx}$$ Example 1: Determine the RMS value of $\boldsymbol{y = 4x^3}$ between $\boldsymbol{x = 0}$ and $\boldsymbol{x = 3}$ $$= \sqrt{\frac{1}{3-0}\int_0^3 (4x^3)^2\,dx}$$ $$= \sqrt{\frac{1}{3}\int_0^3 16x^6\,dx}$$	

Lesson 4: Integral Calculus **Suggested Teaching Time:** 16 hours

Learning Outcome: 3. Be able to use methods of differential and integral calculus to solve engineering problems

Topic	Suggested Teaching	Suggested Resources
	$= \sqrt{\frac{1}{3}\left[\frac{16}{7}x^7\right]_0^3}$ **RMS value** $= 40.8$ Example 2: A current $\boldsymbol{i = 30sin100\pi t}$ **amps** flows in a circuit. Determine its RMS value over a range of $\boldsymbol{t = 0}$ and $\boldsymbol{t = 10\ ms.}$ $\boldsymbol{i^2 = (30sin100\pi t)^2 = 900\sin^2 100\pi t}$ To be able to integrate $\boldsymbol{sin^2 100\pi t}$ we change it to the $\frac{1}{2}(\boldsymbol{1 - cos200\pi t})$ trig identity $\boldsymbol{sin^{2x} = \frac{1}{2}(1 - cos2x)}$ $\boldsymbol{r.m.s. = \sqrt{\frac{1}{10}\int_0^{10} 900 \times \frac{1}{2}(1 - cos200\pi t)\ dt}}$ $\boldsymbol{= \sqrt{\frac{1}{10}\left[450t - \frac{900}{200\pi}sin200\pi t\right]_0^{10}}}$ $\boldsymbol{= \sqrt{\frac{1}{10}\left[4500 - \frac{4.5 \times 0.2756}{\pi}\right]}}$ $\boldsymbol{r.m.s. = 21.21}$	

Lesson 4: Integral Calculus **Suggested Teaching Time:** 16 hours

Learning Outcome: 3. Be able to use methods of differential and integral calculus to solve engineering problems

Topic	Suggested Teaching	Suggested Resources
	Session 7 (3 hrs): Solving differential equations Explain the difference between the *general solution* and the *particular solution* Example 1: Determine the general solution of $$x\frac{dy}{dx} = 3 - 4x^3$$ Rearrange the equation $$\frac{dy}{dx} = \frac{3}{x} - 4x^3$$ Integrating both sides $y = 3\ln x - x^4 + c$ which is the general solution Sample problems Solve the following differential equations: a) $\frac{dy}{dx} = cos3x - 4x$ b) $2x\frac{dy}{dx} = 4 - x^2$	

Lesson 4: Integral Calculus **Suggested Teaching Time:** 16 hours

Learning Outcome: 3. Be able to use methods of differential and integral calculus to solve engineering problems

Topic	Suggested Teaching	Suggested Resources
	Example 2: Determine the particular solution of $$\frac{dy}{dx} = 4 + 3y \text{ given } y = 1 \text{ and } x = 4$$ $$\int dx = \int \frac{dy}{4+3y} \quad let\ u = (4+3y \quad \therefore \frac{du}{dy} = 3$$ $$\int dx = \int \frac{1}{3u} du$$ $$x = \frac{1}{3}\ln(4+3y) + c \quad y = 1 \text{ and } x = 4$$ $$4 = \frac{1}{3} ln\, 7 + c \quad \therefore \quad c = 3.35$$ $$x = \frac{1}{3} ln(4+3y) + 3.35$$ Practice examples: a) Solve the equation $\frac{dy}{dx} = \frac{1x^2}{y}$ given $x = 2$ and $y = 3$ b) Determine the equation of the curve, in terms of y, which satisfies $$x^2 - 1 = xy\frac{dy}{dx}$$ Given the curve passes through $(1, 2)$	

Lesson 4: Integral Calculus **Suggested Teaching Time:** 16 hours

Learning Outcome: 3. Be able to use methods of differential and integral calculus to solve engineering problems

Topic	Suggested Teaching	Suggested Resources
	c) The equation of the bending moment at a point on a simply supported beam is given by $$\frac{dM}{dx} = -W(l - x)$$ **Where** W and x are constants. Find M in terms of x given $$M = \frac{1}{2} \cdot \frac{w}{2} \text{ when } x = 0$$	

Lesson 5: Complex Numbers in Engineering **Suggested Teaching Time:** 15 hours

Learning Outcome: 4. Be able to apply complex numbers and complex analysis to solve engineering problems

Topic	Suggested Teaching	Suggested Resources
AC 4.1 Evaluate complex equations using rectangular and polar forms of complex numbers **AC 4.2** Use **complex function analysis** to obtain solutions to engineering problems.	**Session 1 (3 hrs): Working with complex numbers** Revision of the evaluation of complex numbers. Calculate from the Cartesian form of a complex number the modulus and the argument. Example: For a complex number $\boldsymbol{Z = 5 + j3}$ $\text{Modulus}\lvert \boldsymbol{Z} \rvert = \sqrt{(\boldsymbol{5^2 + 3^{2)}}} = \boldsymbol{5.83}$ $\text{Argument } \boldsymbol{argZ = \theta = tan^{-1}\frac{3}{5} = 31°}$ $\text{Polar form } \boldsymbol{5.83\angle 31°}$ Example: Express each of the following in polar form $\boldsymbol{3 - j3}$ $\boldsymbol{-3 + j3}$ $\boldsymbol{-6 + j}$ Represent complex numbers on an Argand diagram Subtraction and addition of complex numbers	**Book:** Bird. J. O., *Higher Engineering Mathematics* 7th edition (Routledge 2014) ISBN-13: 978-0415662826 **Websites:** http://www.mathcentre.ac.uk/links http://mathworld.wolfram.com/ http://www.mathcentre.ac.uk/

Lesson 5: Complex Numbers in Engineering **Suggested Teaching Time:** 15 hours

Learning Outcome: 4. Be able to apply complex numbers and complex analysis to solve engineering problems

Topic	Suggested Teaching	Suggested Resources
	Example: If $\boldsymbol{Z_1 = (a = jb)}$ and $\boldsymbol{Z_2 = (c + jd)}$ then $\boldsymbol{Z_1 + Z_2 = (a + jb) + (c + jd)}$ $\boldsymbol{= (a + c) + j(b + d)}$ $\boldsymbol{Z_1 - Z_2 = (a + jb) - (c + jd)}$ $\boldsymbol{= (a - c) - j(b - d)}$ Find solutions for each of the following: $\boldsymbol{(3 + j2) + (5 + j6)}$ $\boldsymbol{(1 = j2) - (4 + j)}$ Multiplying complex numbers Find the product of: $\boldsymbol{(3 + j2)(4 + j)}$ $\boldsymbol{= 12 + j3 + j8 + j^2}$ $(j^2 2 = -2 \ because \ j^2 = -1)$ $\boldsymbol{= 12 + j3 + j8 - 2}$ $\boldsymbol{= 10 + j11}$ Example problems: Find the products each of the following: $\boldsymbol{(2 - j6)(3 - j7)}$ $\boldsymbol{-2 + j3)(-5 - j)}$ $\boldsymbol{(3 - j5)(3 - j3)(1 - j)}$ $\boldsymbol{(2 + j)(1 - j)(-3 + j2)}$	

Lesson 5: Complex Numbers in Engineering **Suggested Teaching Time:** 15 hours

Learning Outcome: 4. Be able to apply complex numbers and complex analysis to solve engineering problems

Topic	Suggested Teaching	Suggested Resources
	Session 2 (**2 hrs): Division of complex numbers** Explain that the complex conjugate of a complex number is obtained by changing the sign of the imaginary part i.e. if $x + jy$ represents a complex number then $x - jy$ is known as its conjugate. Example: $(3 - j4)(3 + j4)$ $= 9 + j12 - j12 - j^2 16 \qquad (-[-1] = 16)$ $= 9 + 16 = 25$ The product of a complex number and its complex conjugate can be evaluated on sight $(x + jy)(x - jy) = x^2 = y^2$ Example: $\frac{4 + j5}{1 - j} = \frac{(4 - j5)(1 + j)}{(1 - j)(1 + j)}$ $= \frac{4 + j5 + j4 + j^2 5}{1 - j + j - j2}$ $= \frac{4 + j9 + (-1)5}{1 - (-1)}$ $= \frac{-1}{2} + j\frac{9}{2}$ $= -0.5 + j4.5$	

Lesson 5: Complex Numbers in Engineering **Suggested Teaching Time:** 15 hours

Learning Outcome: 4. Be able to apply complex numbers and complex analysis to solve engineering problems

Topic	Suggested Teaching	Suggested Resources
	Evaluate each of the following a) $\frac{2-j5}{3+j4}$ b) $\frac{7+j3}{8-j3}$ Example 1 The impedance of an electrical circuit having a resistance and inductive reactance in series is given by the complex number $Z = 5 + j6$. Find the admittance Y of a circuit $Y = \frac{1}{Z}$ Example 2 Two impedances Z_1 and Z_2 are denoted by the complex $Z_1 = 1 + j5$ and $Z_2 = j8$. Determine the equivalent impedance Z when a) Z_1 and Z_2 are in series b) Z_1 and Z_2 are in parallel	

Lesson 5: Complex Numbers in Engineering **Suggested Teaching Time:** 15 hours

Learning Outcome: 4. Be able to apply complex numbers and complex analysis to solve engineering problems

Topic	Suggested Teaching	Suggested Resources
	Session 3 (**2 hrs**): **Complex equations** If two complex numbers are equal then their imaginary parts are equal If $a + jb = c + jc$ then $a = c$ and $b = d$ Example 1: Solve $3(x + jy) = 9 - j3$ $3x + j3y = 9 - j3$ Equating real parts $3x = 9 \therefore x = 3$ Equating imaginary parts $3y = 3 \therefore y = 1$ $y = 2 - x \therefore -5x - 4(2 - x) = 3$ Example 2: Solve $(x - j4y) + (y - j5x) = 2 + j3$ $(x + y)j(-4y - 5x) = 2 + j3$ Equating both terms $x + y = 2$ and $5x - 4y = 3$ $y = 2 - x \therefore -5x - 4(2 - x) = 3$ $-5x - 8 + 4x = 3$ $x = -11$	

Lesson 5: Complex Numbers in Engineering **Suggested Teaching Time:** 15 hours

Learning Outcome: 4. Be able to apply complex numbers and complex analysis to solve engineering problems

Topic	Suggested Teaching	Suggested Resources
	(sub $-\mathbf{11}$ into $\boldsymbol{y=2-x}$) $\boldsymbol{y=13}$ Solve each of the following equations a) $(3-j2)(2+j)=a+jb$ b) $(2-j3)=\sqrt{a+jb}$ c) $(x-j4y)+(y-j5x)=2+j3$	
	Session 4 (**3 hrs): Addition and subtraction in polar form** Show that it is not possible to add or subtract directly in polar form. Each complex number must be converted into Cartesian form and then converted back to Polar form. Example: Evaluate in polar form. $5\angle-45°+2\angle30°-4\angle120°$ $5\angle-45°=5(cos-45°+jsin-45°)$ $=5cos-45°+j5sin-45°$ $\underline{3.54-j3.54}$ $2\angle30°=2(cos30°+jsin30°)$	

Lesson 5: Complex Numbers in Engineering **Suggested Teaching Time:** 15 hours

Learning Outcome: 4. Be able to apply complex numbers and complex analysis to solve engineering problems

Topic	Suggested Teaching	Suggested Resources
	$= 2cos30° + j2sin30°$ $4\angle 120° = 4(cos120° + jsin120°)$ $= 4cos120° + j4sin120°$ $= \underline{-2.0 + j3.46}$ $5\angle -45° + 2\angle 30° - 4\angle 120°$ $= (3.54 - j3.54) + (1.73 + j1.0) - (-2.0 + j3.46)$ $= 7.27 - j6.0$ Which is in the 4th quadrant $\sqrt{7.27^2 + 6.2^2}\ \angle \tan^{-1}\frac{6.0}{7.27}$ $= \underline{9.43\angle -39.45°}$ Evaluate in polar form a) $4\angle 30° + 3\angle 22.5°$ b) $5.8\angle 58° + 3\angle 135° - 3\angle -40°$ State that there are many applications of complex numbers particularly in alternating current theory and vector analysis. In ac theory multiplying a phasor by *j* rotates in a positive direction by 90^0 and multiplying a phasor by –*j* rotates it through-90^0	

Lesson 5: Complex Numbers in Engineering **Suggested Teaching Time:** 15 hours

Learning Outcome: 4. Be able to apply complex numbers and complex analysis to solve engineering problems

Topic	Suggested Teaching	Suggested Resources
	Example 1 Determine the resistance and series inductance (capacitance) of the following impedances a) $(6 + j8)\Omega$ Resistance = 6Ω Reactance = $8\Omega = X_L$ Because the imaginary part is positive then the reactance is inductive. The inductance $XL = 2\pi fl$ $\therefore L = \frac{X_L}{2\pi f} = \frac{8}{2\pi 50}$ $= \underline{0.025H}$	

Lesson 5: Complex Numbers in Engineering **Suggested Teaching Time:** 15 hours

Learning Outcome: 4. Be able to apply complex numbers and complex analysis to solve engineering problems

Topic	Suggested Teaching	Suggested Resources
	b) $\boldsymbol{-j5}$ $Z = (0 - j35)$ the resistance $= 0$ and the reactance $= \mathbf{35\Omega}$. The imaginary part is negative so the reactance is capacitive. $\boldsymbol{X_C = 35\Omega}$ and $\boldsymbol{X_C = \frac{1}{2\pi f C}}$ $\boldsymbol{C = \frac{1}{2\pi(50)(35)} \times 10^6}$ $= \underline{\mathbf{90.9\mu F}}$ Example 2 A 250V,50Hz supply is connected across an impedance of $(\mathbf{30 - j50})\mathbf{\Omega}$ Determine the: a) Resistance b) Capacitance c) Magnitude of the impedance and its phase angle d) Current flowing	

Lesson 5: Complex Numbers in Engineering **Suggested Teaching Time:** 15 hours

Learning Outcome: 4. Be able to apply complex numbers and complex analysis to solve engineering problems

Topic	Suggested Teaching	Suggested Resources
	a) Resistance $= \mathbf{30\Omega}$ b) Capacitance $= \frac{\mathbf{1}}{\mathbf{2\pi(50)(50)}} \times \mathbf{10^6}$ $= \mathbf{63.65\mu F}$ c) Impedance $= \lvert\mathbf{Z}\rvert = \sqrt{\mathbf{30^2 + (-50)^2}} = \mathbf{58.3\Omega}$ Phase angle $\mathbf{arg.Z\ tan^{-1}}\left[\frac{\mathbf{-50}}{\mathbf{30}}\right] = \mathbf{59.04°}$ d) Current flowing $= \mathbf{I} = \frac{\mathbf{v}}{\mathbf{Z}} = \frac{\mathbf{230\angle 0°}}{\mathbf{58.3\angle - 59.04}}$ $= \underline{\mathbf{(3.95\angle 059.04°A)}}$	
	Session 5 (3 hrs): De Moivre's Theorem De Moivre's theorem State that the theorem is used to determine powers and roots of complex numbers. $\mathbf{r\angle\theta \times r\angle\theta = r^2\angle 2\theta}$ i.e. $\mathbf{(r\angle\theta)^n = r^n\angle n\theta}$	

Lesson 5: Complex Numbers in Engineering **Suggested Teaching Time:** 15 hours

Learning Outcome: 4. Be able to apply complex numbers and complex analysis to solve engineering problems

Topic	Suggested Teaching	Suggested Resources
	Example 1 Express in polar form $(4\angle 32°)^3 = 4^3\angle 3 \times 32°$ $= \underline{64\angle 96°}$ Example 2 Determine the value of $(6 + j7)$ in polar and rectangular form $-6 + j7 = \sqrt{(-6)^2 + (7)^2}\angle tan^{-1}\frac{7}{-6}$ $= \sqrt{85}\angle 130.6°$ $180° - 49.4°$ 2nd quadrant Using De Moivre's $(-6 + j7)^4 = \left(\sqrt{85}\angle 130.6°\right)^4$ $= \sqrt{85^4}\angle 4 \times 130.6°$ $= 7225\angle 522.4°$ $= 7225\angle 162.4°$ $(522.4 - 360 = 162.4)$ $r\angle\theta = rcos\theta + jrsin\theta$ $7225\angle 162.4° = 7225cos162.4° + j7225sin162.4°$ $= \underline{-6886.8 + j2184.6}$	

Lesson 5: Complex Numbers in Engineering **Suggested Teaching Time:** 15 hours

Learning Outcome: 4. Be able to apply complex numbers and complex analysis to solve engineering problems

Topic	Suggested Teaching	Suggested Resources
	Examples: Express each of the following in polar and rectangular form a) $(\mathbf{5}+\boldsymbol{j}\mathbf{6})^{\mathbf{3}}$ b) $(\mathbf{3}-\boldsymbol{j}\mathbf{8})^{\mathbf{5}}$ c) $(-\mathbf{2}+\boldsymbol{j}\mathbf{7})^{\mathbf{4}}$	
	Session 6 (3 hrs): Roots of complex numbers Show by using De Moivre`s theorem how to determine the square root of a complex number Put $\boldsymbol{n}=\frac{\mathbf{1}}{\mathbf{2}}$ in the theorem: $$\sqrt{\boldsymbol{r}\angle\boldsymbol{\theta}}=(\boldsymbol{r}\angle\boldsymbol{\theta})^{\frac{1}{2}}=\boldsymbol{r}^{\frac{1}{2}}\angle\frac{\mathbf{1}}{\mathbf{2}}\boldsymbol{\theta}=\sqrt{\boldsymbol{r}}\angle\frac{\boldsymbol{\theta}}{\mathbf{2}}$$ It has two equal roots but opposite in sign Example: Determine the two square roots of the complex number $\mathbf{6}+\boldsymbol{j}\mathbf{10}$ in polar and Cartesian forms. $$6+j10=\sqrt{6^2+10^2}\angle\tan^{-1}\frac{10}{6}$$ $$=\underline{\mathbf{11.66}\angle\mathbf{59.04}}°$$	

Lesson 5: Complex Numbers in Engineering **Suggested Teaching Time:** 15 hours

Learning Outcome: 4. Be able to apply complex numbers and complex analysis to solve engineering problems

Topic	Suggested Teaching	Suggested Resources
	Two solutions occur to obtain the second solution the first and second roots have the same modulus but displaced 180^0 from the first root. $\frac{360}{2}$ First root $(6+j10)^{\frac{1}{2}} = \sqrt{11.66}\angle\frac{1}{2}\times 59.04°$ $= \underline{3.42\angle 29.52°}$ Second root $= \sqrt{11.66}\angle\frac{360+59.04}{2}$ $= 3.42\angle 209.52°$ or second root could have been found by $3.42\angle(180+29.52) = 3.42\angle 209.52°$ Cartesian form $3.42\angle 29.52° = 3.42(cos29.52° + jsin29.52°$ $= \underline{2.98+j1.69}$ $3.42\angle 209.52° = 3.42(cos209.52° + jsin209.52°$ $= \underline{-2.98-j1.69}$ Cartesian form $= \underline{\pm(2.98+j1.69)}$	

Lesson 5: Complex Numbers in Engineering **Suggested Teaching Time:** 15 hours

Learning Outcome: 4. Be able to apply complex numbers and complex analysis to solve engineering problems

Topic	Suggested Teaching	Suggested Resources
	Determine the square roots of each of the following complex numbers in Cartesian form a) $\mathbf{3 - j4}$ b) $\mathbf{-1 - j2}$ c) $\mathbf{-6 - j5}$	