Candidates should refer to this 501 formula booklet throughout the course and become familiar with it. Candidates will need to use a clean copy of this document for the 9209-501 exam and for the sample questions.

The online version of the full Mathematical handbook can be found at http://homepage.ntu.edu.tw/~wttsai/MathModel/Mathematical%20Formula%20Handbook.pdf

(Please note we have extracted formula which is not relevant to this unit. The full online version is not required.)
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1 Series

Arithmetic and Geometric progressions

A.P. \[ S_n = a + (a + d) + (a + 2d) + \cdots + [a + (n - 1)d] = \frac{n}{2}[2a + (n - 1)d] \]

G.P. \[ S_n = a + ar + ar^2 + \cdots + ar^{n-1} = a \frac{1 - r^n}{1 - r}, \quad (S_\infty = \frac{a}{1 - r} \text{ for } |r| < 1) \]

(These results also hold for complex series.)

Binomial expansion

\[
(1 + x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots
\]

If \( n \) is a positive integer the series terminates and is valid for all \( x \): the term in \( x^r \) is \( \binom{n}{r} x^r \) or \( \frac{n!}{r!(n-r)!} \) is the number of different ways in which an unordered sample of \( r \) objects can be selected from a set of \( n \) objects without replacement. When \( n \) is not a positive integer, the series does not terminate: the infinite series is convergent for \( |x| < 1 \).

Taylor and Maclaurin Series

If \( y(x) \) is well-behaved in the vicinity of \( x = a \) then it has a Taylor series,

\[
y(x) = y(a) + \frac{dy}{dx} \cdot (x - a) + \frac{d^2y}{dx^2} \cdot \frac{(x - a)^2}{2!} + \frac{d^3y}{dx^3} \cdot \frac{(x - a)^3}{3!} + \cdots
\]

where \( u = x - a \) and the differential coefficients are evaluated at \( x = a \). A Maclaurin series is a Taylor series with \( a = 0 \),

\[
y(x) = y(0) + \frac{dy}{dx} \cdot x + \frac{d^2y}{dx^2} \cdot \frac{x^2}{2!} + \frac{d^3y}{dx^3} \cdot \frac{x^3}{3!} + \cdots
\]

Power series with real variables

\[
e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots \quad \text{valid for all } x
\]

\[
\ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} + \cdots + (-1)^{n+1}\frac{x^n}{n} + \cdots \quad \text{valid for } -1 < x \leq 1
\]

\[
\cos x = \frac{e^{ix} + e^{-ix}}{2} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \quad \text{valid for all values of } x
\]

\[
\sin x = \frac{e^{ix} - e^{-ix}}{2i} = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots \quad \text{valid for all values of } x
\]

\[
\tan x = x + \frac{x^3}{3} + \frac{2}{15}x^5 + \cdots \quad \text{valid for } -\frac{\pi}{2} < x < \frac{\pi}{2}
\]

\[
\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots \quad \text{valid for } -1 \leq x \leq 1
\]

\[
\sin^{-1} x = x + \frac{1}{2}\frac{x^3}{3} + \frac{13}{24}\frac{x^5}{5} + \cdots \quad \text{valid for } -1 < x < 1
\]
**Integer series**

\[ \sum_{i=1}^{N} n = 1 + 2 + 3 + \cdots + N = \frac{N(N + 1)}{2} \]

\[ \sum_{i=1}^{N} n^2 = 1^2 + 2^2 + 3^2 + \cdots + N^2 = \frac{N(N + 1)(2N + 1)}{6} \]

\[ \sum_{i=1}^{N} n^3 = 1^3 + 2^3 + 3^3 + \cdots + N^3 = \left(1 + 2 + 3 + \cdots + N\right)^2 = \frac{N^2(N + 1)^2}{4} \]

\[ \sum_{i=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \ln 2 \quad \text{[see expansion of } \ln(1 + x)\text{]} \]

\[ \sum_{i=1}^{\infty} \frac{(-1)^{n+1}}{2n - 1} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \frac{\pi}{4} \quad \text{[see expansion of } \tan^{-1} x\text{]} \]

\[ \sum_{i=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots = \frac{\pi^2}{6} \]

\[ \sum_{i=1}^{N} n(n + 1)(n + 2) = 1.2.3 + 2.3.4 + \cdots + N(N + 1)(N + 2) = \frac{N(N + 1)(N + 2)(N + 3)}{4} \]

This last result is a special case of the more general formula,

\[ \sum_{i=1}^{N} n(n + 1)(n + 2) \cdots (n + r) = \frac{N(N + 1)(N + 2) \cdots (N + r)(N + r + 1)}{r + 2} \]
2 Vector Algebra

If \( i, j, k \) are orthonormal vectors and \( A = A_x i + A_y j + A_z k \) then \( |A|^2 = A_x^2 + A_y^2 + A_z^2 \). [Orthonormal vectors \( \equiv \) orthogonal unit vectors.]

**Scalar product**

\[
A \cdot B = |A| |B| \cos \theta
\]

\[
= A_x B_x + A_y B_y + A_z B_z = [ A_x A_y A_z ] [ B_x \ B_y \ B_z ]
\]

Scalar multiplication is commutative: \( A \cdot B = B \cdot A \).

**Equation of a line**

A point \( r \equiv (x, y, z) \) lies on a line passing through a point \( a \) and parallel to vector \( b \) if

\[
r = a + \lambda b
\]

with \( \lambda \) a real number.

**Equation of a plane**

A point \( r \equiv (x, y, z) \) is on a plane if either

(a) \( r \cdot \hat{d} = |d| \), where \( d \) is the normal from the origin to the plane, or

(b) \( \frac{x}{X} + \frac{y}{Y} + \frac{z}{Z} = 1 \) where \( X, Y, Z \) are the intercepts on the axes.

**Vector product**

\( A \times B = n |A| |B| \sin \theta \), where \( \theta \) is the angle between the vectors and \( n \) is a unit vector normal to the plane containing \( A \) and \( B \) in the direction for which \( A, B, n \) form a right-handed set of axes.

\[
A \times B =\begin{vmatrix}
i & j & k \\
A_x & A_y & A_z \\
B_x & B_y & B_z \\
\end{vmatrix}
\]

\[
A \times B = \begin{bmatrix}
0 & -A_z & A_y \\
A_z & 0 & -A_x \\
-A_y & A_x & 0
\end{bmatrix}
\begin{bmatrix}
B_x \\
B_y \\
B_z
\end{bmatrix}
\]

Vector multiplication is not commutative: \( A \times B = -B \times A \).
3 Matrix Algebra

Unit matrices

The unit matrix $I$ of order $n$ is a square matrix with all diagonal elements equal to one and all off-diagonal elements zero, i.e., $(I)_{ij} = \delta_{ij}$. If $A$ is a square matrix of order $n$, then $AI = IA = A$. Also $I = I^{-1}$.

$I$ is sometimes written as $I_n$ if the order needs to be stated explicitly.

Products

If $A$ is a $(n \times l)$ matrix and $B$ is a $(l \times m)$ then the product $AB$ is defined by

$$(AB)_{ij} = \sum_{k=1}^{l} A_{ik}B_{kj}$$

In general $AB \neq BA$.

Transpose matrices

If $A$ is a matrix, then transpose matrix $A^T$ is such that $(A^T)_{ij} = (A)_{ji}$.

Inverse matrices

If $A$ is a square matrix with non-zero determinant, then its inverse $A^{-1}$ is such that $AA^{-1} = A^{-1}A = I$.

$$(A^{-1})_{ij} = \frac{\text{transpose of cofactor of } A_{ij}}{|A|}$$

where the cofactor of $A_{ij}$ is $(-1)^{i+j}$ times the determinant of the matrix $A$ with the $j$-th row and $i$-th column deleted.

Determinants

If $A$ is a square matrix then the determinant of $A$, $|A| \equiv \det A$ is defined by

$$|A| = \sum_{i,j,k,...} \varepsilon_{ijk...} A_{i1}A_{j2}A_{k3}...$$

where the number of the suffixes is equal to the order of the matrix.

$2 \times 2$ matrices

If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then,

$$|A| = ad - bc, \quad A^T = \begin{pmatrix} a & c \\ b & d \end{pmatrix}, \quad A^{-1} = \frac{1}{|A|} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Product rules

$$(AB...N)^T = N^T...B^T A^T$$

$$(AB...N)^{-1} = N^{-1}...B^{-1} A^{-1}$$  \hspace{1cm} \text{(if individual inverses exist)}$$

$$|AB...N| = |A| \cdot |B| \cdot \ldots \cdot |N|$$  \hspace{1cm} \text{(if individual matrices are square)}$

Orthogonal matrices

An orthogonal matrix $Q$ is a square matrix whose columns $q_i$ form a set of orthonormal vectors. For any orthogonal matrix $Q$,

$$Q^{-1} = Q^T, \quad |Q| = \pm 1, \quad Q^T \text{ is also orthogonal.}$$
Solving sets of linear simultaneous equations

If \( A \) is square then \( Ax = b \) has a unique solution \( x = A^{-1}b \) if \( A^{-1} \) exists, i.e., if \( |A| \neq 0 \).
If \( A \) is square then \( Ax = 0 \) has a non-trivial solution if and only if \( |A| = 0 \).
An over-constrained set of equations \( Ax = b \) is one in which \( A \) has \( m \) rows and \( n \) columns, where \( m \) (the number of equations) is greater than \( n \) (the number of variables). The best solution \( x \) (in the sense that it minimizes the error \( |Ax - b| \)) is the solution of the \( n \) equations \( A^T Ax = A^T b \). If the columns of \( A \) are orthonormal vectors then \( x = A^T b \).

Eigenvalues and eigenvectors

The \( n \) eigenvalues \( \lambda_i \) and eigenvectors \( u_i \) of an \( n \times n \) matrix \( A \) are the solutions of the equation \( Au = \lambda u \). The eigenvalues are the zeros of the polynomial of degree \( n \), \( P_n(\lambda) = |A - \lambda I| \). If \( A \) is Hermitian then the eigenvalues \( \lambda_i \) are real and the eigenvectors \( u_i \) are mutually orthogonal. \( |A - \lambda I| = 0 \) is called the characteristic equation of the matrix \( A \).

\[
\text{Tr } A = \sum \lambda_i, \quad \text{also } |A| = \prod \lambda_i.
\]

If \( S \) is a symmetric matrix, \( A \) is the diagonal matrix whose diagonal elements are the eigenvalues of \( S \), and \( U \) is the matrix whose columns are the normalized eigenvectors of \( A \), then

\[
U^T SU = A \quad \text{and} \quad S = U A U^T.
\]

If \( x \) is an approximation to an eigenvector of \( A \) then \( x^T Ax/(x^T x) \) (Rayleigh's quotient) is an approximation to the corresponding eigenvalue.

Commutators

\[
\begin{align*}
[A, B] & \equiv AB - BA \\
[A, B] & = -[B, A] \\
[A, B]^\dagger & = [B^\dagger, A^\dagger] \\
[A, [B, C]] + [B, [C, A]] + [C, [A, B]] & = 0
\end{align*}
\]
4 Vector Calculus

Notation

$\phi$ is a scalar function of a set of position coordinates. In Cartesian coordinates $\phi = \phi(x, y, z)$; in cylindrical polar coordinates $\phi = \phi(\rho, \theta, z)$; in spherical polar coordinates $\phi = \phi(r, \theta, \varphi)$; in cases with radial symmetry $\phi = \phi(r)$. $A$ is a vector function whose components are scalar functions of the position coordinates: in Cartesian coordinates $A = iA_x + jA_y + kA_z$, where $A_x, A_y, A_z$ are independent functions of $x, y, z$.

In Cartesian coordinates $\nabla \ (\text{`del'}) \equiv \frac{\partial}{\partial x} + j \frac{\partial}{\partial y} + k \frac{\partial}{\partial z} \equiv \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix}$

$\nabla \phi = \nabla \phi$, $\text{div} \ A = \nabla \cdot A$, $\text{curl} \ A = \nabla \times A$

Identities

$\nabla (\phi_1 + \phi_2) \equiv \nabla \phi_1 + \nabla \phi_2$,  $\text{div} \ (A_1 + A_2) \equiv \text{div} \ A_1 + \text{div} \ A_2$  
$\nabla (\phi_1 \phi_2) \equiv \phi_1 \nabla \phi_2 + \phi_2 \nabla \phi_1$,  $\text{curl} \ (A_1 + A_2) \equiv \text{curl} \ A_1 + \text{curl} \ A_2$  
$\text{div} \ (\phi A) \equiv \phi \text{div} \ A + (\nabla \phi) \cdot A$,  $\text{curl} \ (\phi A) \equiv \phi \text{curl} \ A + (\nabla \phi) \times A$  
$\text{div} \ (A_1 \times A_2) \equiv A_2 \cdot \text{curl} \ A_1 - A_1 \cdot \text{curl} \ A_2$  
$\text{curl} \ (A_1 \times A_2) \equiv A_1 \text{div} \ A_2 - A_2 \text{div} \ A_1 + (A_2 \cdot \nabla) A_1 - (A_1 \cdot \nabla) A_2$  
$\text{div} \ (\text{curl} \ A) \equiv 0$,  $\text{curl} \ (\text{grad} \ \phi) \equiv 0$  
$\text{curl} \ (\text{curl} \ A) \equiv \text{grad} \ (\text{div} \ A) - \nabla^2 A$  
$\nabla \cdot (A_1 \cdot A_2) \equiv A_1 \cdot (\text{curl} \ A_2) + (A_1 \cdot \nabla) A_2 + A_2 \cdot (\text{curl} \ A_1) + (A_2 \cdot \nabla) A_1$
## Grad, Div, Curl and the Laplacian

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<td><strong>Conversion to</strong></td>
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<tr>
<td>Cartesian Coordinates</td>
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<tr>
<td><strong>Vector A</strong></td>
<td>( A_x i + A_y j + A_z k )</td>
<td>( A_\rho \hat{\rho} + A_\theta \hat{\theta} + A_\phi \hat{\phi} )</td>
<td>( A_r \hat{r} + A_\theta \hat{\theta} + A_\phi \hat{\phi} )</td>
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<tr>
<td><strong>Gradient ( \nabla \phi )</strong></td>
<td>( \frac{\partial \phi}{\partial x} i + \frac{\partial \phi}{\partial y} j + \frac{\partial \phi}{\partial z} k )</td>
<td>( \frac{\partial \phi}{\partial \rho} \hat{\rho} + \frac{1}{\rho} \frac{\partial \phi}{\partial \theta} \hat{\theta} + \frac{1}{\rho \sin \theta} \frac{\partial \phi}{\partial \phi} \hat{\phi} )</td>
<td>( \frac{\partial \phi}{\partial r} \hat{r} + \frac{1}{r} \frac{\partial \phi}{\partial \theta} \hat{\theta} + \frac{1}{r \sin \theta} \frac{\partial \phi}{\partial \phi} \hat{\phi} )</td>
</tr>
<tr>
<td><strong>Divergence ( \nabla \cdot A )</strong></td>
<td>( \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} )</td>
<td>( \frac{1}{\rho} \frac{\partial (\rho A_\rho)}{\partial \rho} + \frac{1}{\rho} \frac{\partial A_\theta}{\partial \theta} + \frac{\partial A_\phi}{\partial z} )</td>
<td>( \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 A_r) + \frac{1}{r} \frac{\partial A_\theta}{\partial \theta} + \frac{1}{r \sin \theta} \frac{\partial A_\phi}{\partial \phi} )</td>
</tr>
<tr>
<td><strong>Curl ( \nabla \times A )</strong></td>
<td>( \begin{vmatrix} i &amp; j &amp; k \ \frac{\partial}{\partial x} &amp; \frac{\partial}{\partial y} &amp; \frac{\partial}{\partial z} \ A_x &amp; A_y &amp; A_z \end{vmatrix} )</td>
<td>( \begin{vmatrix} 1/\rho &amp; \hat{\phi} &amp; 1/\rho \ \partial &amp; \partial &amp; \partial \ A_\rho &amp; \rho A_\phi &amp; A_z \end{vmatrix} )</td>
<td>( \begin{vmatrix} 1/r^2 &amp; \hat{\theta} &amp; 1/\rho \ \partial &amp; \partial &amp; \partial \ rA_\theta &amp; rA_\phi &amp; rA_\phi \sin \theta \end{vmatrix} )</td>
</tr>
<tr>
<td><strong>Laplacian ( \nabla^2 \phi )</strong></td>
<td>( \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} )</td>
<td>( \frac{1}{\rho} \frac{\partial}{\partial \rho} \left( \rho \frac{\partial \phi}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2 \phi}{\partial \phi^2} + \frac{\partial^2 \phi}{\partial z^2} )</td>
<td>( \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial \phi}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial \phi}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 \phi}{\partial \phi^2} )</td>
</tr>
</tbody>
</table>

\( x = r \cos \theta \quad y = r \sin \theta \quad z = z \)
5 Complex Variables

Complex numbers

The complex number $z = x + iy = r(\cos \theta + i \sin \theta) = re^{i(\theta + 2\pi n)}$, where $i^2 = -1$ and $n$ is an arbitrary integer. The real quantity $r$ is the modulus of $z$ and the angle $\theta$ is the argument of $z$. The complex conjugate of $z$ is $z^* = x - iy = r(\cos \theta - i \sin \theta) = re^{-i\theta}$, $zz^* = |z|^2 = x^2 + y^2$

De Moivre’s theorem

$(\cos \theta + i \sin \theta)^n = e^{in\theta} = \cos n\theta + i \sin n\theta$

Power series for complex variables.

\[
e^z = 1 + z + \frac{z^2}{2!} + \cdots + \frac{z^n}{n!} + \cdots \quad \text{convergent for all finite } z
\]
\[
\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots \quad \text{convergent for all finite } z
\]
\[
\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots \quad \text{convergent for all finite } z
\]
\[
\ln(1 + z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \cdots \quad \text{principal value of } \ln(1 + z)
\]

This last series converges both on and within the circle $|z| = 1$ except at the point $z = -1$.

\[
\tan^{-1} z = z - \frac{z^3}{3} + \frac{z^5}{5} - \cdots
\]

This last series converges both on and within the circle $|z| = 1$ except at the points $z = \pm i$.

\[
(1 + z)^n = 1 + nz + \frac{n(n-1)}{2!}z^2 + \frac{n(n-1)(n-2)}{3!}z^3 + \cdots
\]

This last series converges both on and within the circle $|z| = 1$ except at the point $z = -1$.
6 Trigonometric Formulae

\[
\begin{align*}
\cos^2 A + \sin^2 A &= 1, & \sec^2 A - \tan^2 A &= 1, & \csc^2 A - \cot^2 A &= 1, \\
\sin 2A &= 2 \sin A \cos A, & \cos 2A &= \cos^2 A - \sin^2 A, & \tan 2A &= \frac{2 \tan A}{1 - \tan^2 A}.
\end{align*}
\]

\[
\begin{align*}
\sin (A \pm B) &= \sin A \cos B \pm \cos A \sin B, & \cos (A \pm B) &= \cos A \cos B \mp \sin A \sin B, & \tan (A \pm B) &= \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}, \\
\sin A \cos B &= \frac{\cos (A + B) + \cos (A - B)}{2}, & \cos A \sin B &= \frac{\cos (A - B) - \cos (A + B)}{2}, & \sin A \cos B &= \frac{\sin (A + B) + \sin (A - B)}{2}.
\end{align*}
\]

\[
\begin{align*}
\sin A + \sin B &= 2 \sin \frac{A + B}{2} \cos \frac{A - B}{2}, & \cos^2 A &= \frac{1 + \cos 2A}{2}, \\
\sin A - \sin B &= 2 \cos \frac{A + B}{2} \sin \frac{A - B}{2}, & \sin^2 A &= \frac{1 - \cos 2A}{2}, \\
\cos A + \cos B &= 2 \cos \frac{A + B}{2} \cos \frac{A - B}{2}, & \cos^3 A &= \frac{3 \cos A + \cos 3A}{4}, \\
\cos A - \cos B &= -2 \sin \frac{A + B}{2} \sin \frac{A - B}{2}, & \sin^3 A &= \frac{3 \sin A - \sin 3A}{4},
\end{align*}
\]

Relations between sides and angles of any plane triangle

In a plane triangle with angles \(A, B, C\) and sides opposite \(a, b, c\) respectively,

\[
\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = \text{diameter of circumscribed circle}.
\]

\[
a^2 = b^2 + c^2 - 2bc \cos A,
\]

\[
a = b \cos C + c \cos B,
\]

\[
\cos A = \frac{b^2 + c^2 - a^2}{2bc},
\]

\[
\tan \frac{A - B}{2} = \frac{a - b}{a + b} \cot \frac{C}{2},
\]

\[
\text{area} = \frac{1}{2}ab \sin C = \frac{1}{2}bc \sin A = \frac{1}{2}ca \sin B = \sqrt{s(s - a)(s - b)(s - c)}, \quad \text{where} \ s = \frac{1}{2}(a + b + c).
\]
7 Hyperbolic Functions

\[
\cosh x = \frac{1}{2} (e^x + e^{-x}) = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots \quad \text{valid for all } x
\]
\[
\sinh x = \frac{1}{2} (e^x - e^{-x}) = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots \quad \text{valid for all } x
\]
\[
\cosh ix = \cos x \\
\sinh ix = i \sin x \\
\tanh x = \frac{\sinh x}{\cosh x} \\
\coth x = \frac{\cosh x}{\sinh x} \\
\cosh^2 x - \sinh^2 x = 1
\]

For large positive \( x \):
\[
\cosh x \approx \sinh x \to \frac{e^x}{2} \\
\tanh x \to 1
\]

For large negative \( x \):
\[
\cosh x \approx -\sinh x \to \frac{e^{-x}}{2} \\
\tanh x \to -1
\]

Relations of the functions

\[
\sinh x = -\sinh(-x) \\
\cosh x = \cosh(-x) \\
\tanh x = -\tanh(-x) \\
\sinh x = \frac{2 \tanh (x/2)}{1 - \tanh^2 (x/2)} = \frac{\tanh x}{\sqrt{1 - \tanh^2 x}} \\
\tanh x = \sqrt{1 - \text{sech}^2 x} \\
\coth x = \sqrt{\text{coth}^2 x + 1} \\
\sinh(x/2) = \sqrt{\frac{\cosh x - 1}{2}} \\
\tanh(x/2) = \frac{\cosh x - 1}{\sinh x} = \frac{\sinh x}{\cosh x + 1}
\]

\[
\sinh(2x) = 2 \sinh x \cosh x \\
\cosh(2x) = \cosh^2 x + \sinh^2 x = 2 \cosh^2 x - 1 = 1 + 2 \sinh^2 x \\
\sinh(3x) = 3 \sinh x + 4 \sinh^3 x \\
\cosh(3x) = 4 \cosh^3 x - 3 \cosh x
\]

\[
\tanh(2x) = \frac{2 \tanh x}{1 + \tanh^2 x} \\
\tanh(3x) = \frac{3 \tanh x + \tanh^3 x}{1 + 3 \tanh^2 x}
\]
\[
\sinh(x \pm y) = \sinh x \cosh y \pm \cosh x \sinh y \\
\cosh(x \pm y) = \cosh x \cosh y \pm \sinh x \sinh y \\
\tanh(x \pm y) = \frac{\tanh x \pm \tanh y}{1 \pm \tanh x \tanh y}
\]

\[
\sinh x + \sinh y = 2 \sinh \frac{1}{2}(x + y) \cosh \frac{1}{2}(x - y) \\
\cosh x + \cosh y = 2 \cosh \frac{1}{2}(x + y) \cosh \frac{1}{2}(x - y) \\
\sinh x - \sinh y = 2 \cosh \frac{1}{2}(x + y) \sinh \frac{1}{2}(x - y) \\
\cosh x - \cosh y = 2 \sinh \frac{1}{2}(x + y) \sinh \frac{1}{2}(x - y)
\]

\[
\sinh x \pm \cosh x = \frac{1 \pm \tanh (x/2)}{1 \mp \tanh (x/2)} = e^{\pm x}
\]

\[
\tanh x \pm \tanh y = \frac{\sinh(x \pm y)}{\cosh x \cosh y}
\]

\[
\coth x \pm \coth y = \pm \frac{\sinh(x \pm y)}{\sinh x \sinh y}
\]

**Inverse functions**

\[
\sinh^{-1} \frac{x}{a} = \ln \left( \frac{x + \sqrt{x^2 + a^2}}{a} \right) \quad \text{for } -\infty < x < \infty
\]

\[
\cosh^{-1} \frac{x}{a} = \ln \left( \frac{x + \sqrt{x^2 - a^2}}{a} \right) \quad \text{for } x \geq a
\]

\[
\tanh^{-1} \frac{x}{a} = \frac{1}{2} \ln \left( \frac{a + x}{a - x} \right) \quad \text{for } x^2 < a^2
\]

\[
\coth^{-1} \frac{x}{a} = \frac{1}{2} \ln \left( \frac{x + a}{x - a} \right) \quad \text{for } x^2 > a^2
\]

\[
\sech^{-1} \frac{x}{a} = \ln \left( \frac{a}{x} + \sqrt{\frac{a^2}{x^2} - 1} \right) \quad \text{for } 0 < x \leq a
\]

\[
\cosech^{-1} \frac{x}{a} = \ln \left( \frac{a}{x} + \sqrt{\frac{a^2}{x^2} + 1} \right) \quad \text{for } x \neq 0
\]
8 Limits

\( n^c x^n \to 0 \text{ as } n \to \infty \) if \(|x| < 1\) (any fixed \(c\))

\( x^n/n! \to 0 \text{ as } n \to \infty \) (any fixed \(x\))

\( (1 + x/n)^n \to e^x \text{ as } n \to \infty, \ x \ln x \to 0 \text{ as } x \to 0 \)

If \( f(a) = g(a) = 0 \) then \( \lim_{x \to a} \frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)} \) (l’Hôpital’s rule)
9 Differentiation

\[(uv)' = u'v + uv', \quad \left( \frac{u'}{v} \right)' = \frac{u'v - uv'}{v^2} \]

\[(uv)^{(n)} = u^{(n)}v + nu^{(n-1)}v^{(1)} + \cdots + n \binom{n}{r} u^{(n-r)}v^{(r)} + \cdots + uv^{(n)} \]

where \(\binom{n}{r} = \frac{n!}{r!(n-r)!}\)

Leibniz Theorem

\[
\begin{align*}
\frac{d}{dx} (\sin x) &= \cos x \\
\frac{d}{dx} (\cos x) &= -\sin x \\
\frac{d}{dx} (\tan x) &= \sec^2 x \\
\frac{d}{dx} (\sec x) &= \sec x \tan x \\
\frac{d}{dx} (\cot x) &= -\cosec^2 x \\
\frac{d}{dx} (\cosec x) &= -\cosec x \cot x \\
\frac{d}{dx} (\sinh x) &= \cosh x \\
\frac{d}{dx} (\cosh x) &= \sinh x \\
\frac{d}{dx} (\tanh x) &= \sech^2 x \\
\frac{d}{dx} (\sech x) &= -\sech x \tanh x \\
\frac{d}{dx} (\coth x) &= -\cosech^2 x \\
\frac{d}{dx} (\cosech x) &= -\cosech x \coth x 
\end{align*}
\]
10 Integration

Standard forms

\[ \int x^n \, dx = \frac{x^{n+1}}{n+1} + c \quad \text{for } n \neq -1 \]

\[ \int \frac{1}{x} \, dx = \ln x + c \]

\[ \int e^{ax} \, dx = \frac{1}{a} e^{ax} + c \]

\[ \int x \ln x \, dx = \frac{x^2}{2} \left( \ln x - \frac{1}{2} \right) + c \]

\[ \int \frac{1}{a^2 + x^2} \, dx = \frac{1}{a} \tan^{-1} \left( \frac{x}{a} \right) + c \]

\[ \int \frac{1}{a^2 - x^2} \, dx = \frac{1}{2a} \tanh^{-1} \left( \frac{x}{a} \right) + c = \frac{1}{2a} \ln \left( \frac{a+x}{a-x} \right) + c \quad \text{for } x^2 < a^2 \]

\[ \int \frac{1}{x^2 - a^2} \, dx = -\frac{1}{a} \coth^{-1} \left( \frac{x}{a} \right) + c = \frac{1}{2a} \ln \left( \frac{x-a}{x+a} \right) + c \quad \text{for } x^2 > a^2 \]

\[ \int \frac{x}{(x^2 \pm a^2)^n} \, dx = \frac{-1}{2(n-1)} \frac{1}{(x^2 \pm a^2)^{n-1}} + c \quad \text{for } n \neq 1 \]

\[ \int \frac{x}{x^2 \pm a^2} \, dx = \frac{1}{2} \ln(x^2 \pm a^2) + c \]

\[ \int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \sin^{-1} \left( \frac{x}{a} \right) + c \]

\[ \int \frac{1}{\sqrt{x^2 \pm a^2}} \, dx = \ln \left( x + \sqrt{x^2 \pm a^2} \right) + c \]

\[ \int \frac{x}{\sqrt{x^2 \pm a^2}} \, dx = \sqrt{x^2 \pm a^2} + c \]

\[ \int \sqrt{a^2 - x^2} \, dx = \frac{1}{2} \left[ x \sqrt{a^2 - x^2} + a^2 \sin^{-1} \left( \frac{x}{a} \right) \right] + c \]
\[
\int_0^\infty \frac{1}{(1 + x)^p} \, dx = \pi \csc p \pi 
\]
\[
\int_0^\infty \cos(x^2) \, dx = \int_0^\infty \sin(x^2) \, dx = \frac{1}{\sqrt{2}} \sqrt{\frac{\pi}{2}} 
\]
\[
\int_{-\infty}^{\infty} \exp(-x^2/2\sigma^2) \, dx = \sigma \sqrt{2\pi} 
\]
\[
\int_{-\infty}^{\infty} x^n \exp(-x^2/2\sigma^2) \, dx = \begin{cases} 1 \times 3 \times 5 \times \cdots (n-1)\sigma^{n+1}\sqrt{2\pi} & \text{for } n \geq 2 \text{ and even} \\ 0 & \text{for } n \geq 1 \text{ and odd} \end{cases} 
\]
\[
\int \sin x \, dx = -\cos x + c \\
\int \cos x \, dx = \sin x + c \\
\int \tan x \, dx = -\ln(\cos x) + c \\
\int \sec x \, dx = \ln(|\sec x + \tan x|) + c \\
\int \csc x \, dx = \ln|\csc x - \cot x| + c \\
\int \cot x \, dx = \ln(|\sin x|) + c \\
\int \sin mx \sin nx \, dx = \frac{\sin(m-n)x}{2(m-n)} - \frac{\sin(m+n)x}{2(m+n)} + c \\
\int \cos mx \cos nx \, dx = \frac{\sin(m-n)x}{2(m-n)} + \frac{\sin(m+n)x}{2(m+n)} + c 
\]

**Standard substitutions**

If the integrand is a function of:

- \((a^2 - x^2)\) or \(\sqrt{a^2 - x^2}\)
  
  \(x = a \sin \theta \) or \(x = a \cos \theta \)

- \((x^2 + a^2)\) or \(\sqrt{x^2 + a^2}\)
  
  \(x = a \tan \theta \) or \(x = a \sec \theta \)

- \((x^2 - a^2)\) or \(\sqrt{x^2 - a^2}\)
  
  \(x = a \sec \theta \) or \(x = a \cosh \theta \)

If the integrand is a rational function of \(\sin x\) or \(\cos x\) or both, substitute \(t = \tan(x/2)\) and use the results:

\[
\sin x = \frac{2t}{1 + t^2} \quad \cos x = \frac{1 - t^2}{1 + t^2} \quad dx = \frac{2 \, dt}{1 + t^2}. 
\]

If the integrand is of the form:

\[
\frac{dx}{(ax + b)\sqrt{px + q}} \quad px + q = u^2 \\
\frac{dx}{(ax + b)\sqrt{px^2 + qx + r}} \quad ax + b = \frac{1}{u}. 
\]
Integration by parts
\[ \int_a^b u \, dv = uv \bigg|_a^b - \int_a^b v \, du \]

Differentiation of an integral
If \( f(x, \alpha) \) is a function of \( x \) containing a parameter \( \alpha \) and the limits of integration \( a \) and \( b \) are functions of \( \alpha \) then
\[ \frac{d}{d\alpha} \int_{a(\alpha)}^{b(\alpha)} f(x, \alpha) \, dx = f(b, \alpha) \frac{db}{d\alpha} - f(a, \alpha) \frac{da}{d\alpha} + \int_{a(\alpha)}^{b(\alpha)} \frac{\partial}{\partial \alpha} f(x, \alpha) \, dx. \]

Special case,
\[ \frac{d}{dx} \int_a^x f(y) \, dy = f(x). \]

Dirac \( \delta \)-function
\[ \delta(t - \tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp[i\omega(t - \tau)] \, d\omega. \]

If \( f(t) \) is an arbitrary function of \( t \) then \( \int_{-\infty}^{\infty} \delta(t - \tau)f(t) \, dt = f(\tau). \)
\( \delta(t) = 0 \) if \( t \neq 0 \), also \( \int_{-\infty}^{\infty} \delta(t) \, dt = 1 \)

Reduction formulae

Factorials
\[ n! = n(n-1)(n-2) \ldots 1, \quad 0! = 1. \]

Stirling’s formula for large \( n \):
\[ \ln(n!) \approx n \ln n - n. \]

For any \( p > -1 \),
\[ \int_0^\infty x^p e^{-x} \, dx = p \int_0^\infty x^{p-1} e^{-x} \, dx = p!, \quad (-\frac{1}{2})! = \sqrt{\pi}, \quad \left(\frac{1}{2}\right)! = \sqrt{\pi}/2, \text{ etc.} \]

For any \( p, q > -1 \),
\[ \int_0^1 x^p (1-x)^q \, dx = \frac{p!q!}{(p+q+1)!}. \]

Trigonometrical

If \( m, n \) are integers,
\[ \int_0^{\pi/2} \sin^m \theta \cos^n \theta \, d\theta = \frac{m-1}{m+n} \int_0^{\pi/2} \sin^{m-2} \theta \cos^n \theta \, d\theta = \frac{n-1}{m+n} \int_0^{\pi/2} \sin^m \theta \cos^{n-2} \theta \, d\theta \]

and can therefore be reduced eventually to one of the following integrals
\[ \int_0^{\pi/2} \sin \theta \cos \theta \, d\theta = \frac{1}{2}, \quad \int_0^{\pi/2} \sin \theta \, d\theta = 1, \quad \int_0^{\pi/2} \cos \theta \, d\theta = 1, \quad \int_0^{\pi/2} d\theta = \frac{\pi}{2}. \]

Other
If \( I_n = \int_0^\infty x^n \exp(-ax^2) \, dx \) then
\[ I_n = \frac{(n-1)}{2a} I_{n-2}, \quad I_0 = \frac{1}{2\sqrt{\pi}a}, \quad I_1 = \frac{1}{2a}. \]
11 Differential Equations

Diffusion (conduction) equation
\[
\frac{\partial \psi}{\partial t} = \kappa \nabla^2 \psi
\]

Wave equation
\[
\nabla^2 \psi = \frac{1}{c^2} \frac{\partial^2 \psi}{\partial t^2}
\]

Bessel’s equation
\[
x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} + (x^2 - m^2) y = 0,
\]
solutions of which are Bessel functions \(J_m(x)\) of order \(m\).

Series form of Bessel functions of the first kind
\[
J_m(x) = \sum_{k=0}^{\infty} \frac{(-1)^k (x/2)^{m+2k}}{k!(m+k)!} \quad \text{(integer } m\text{)}.
\]
The same general form holds for non-integer \(m > 0\).
Laplace’s equation

\( \nabla^2 u = 0 \)

If expressed in two-dimensional polar coordinates (see section 4), a solution is

\[ u(\rho, \varphi) = \left[A \rho^n + B \rho^{-n}\right] \left[C \exp(in\varphi) + D \exp(-in\varphi)\right] \]

where \(A, B, C, D\) are constants and \(n\) is a real integer.

If expressed in three-dimensional polar coordinates (see section 4) a solution is

\[ u(r, \theta, \varphi) = \left[A r^l + B r^{-(l+1)}\right] P_l^m \left[C \sin m\varphi + D \cos m\varphi\right] \]

where \(l\) and \(m\) are integers with \(l \geq |m| \geq 0\); \(A, B, C, D\) are constants;

\[ P_l^m(\cos \theta) = \sin |m| \theta \left[ \frac{d}{d(\cos \theta)} \right] |m| P_l(\cos \theta) \]

is the associated Legendre polynomial.

\[ P_l^0(1) = 1. \]

If expressed in cylindrical polar coordinates (see section 4), a solution is

\[ u(\rho, \varphi, z) = J_m(n\rho) \left[A \cos m\varphi + B \sin m\varphi\right] \left[C \exp(nz) + D \exp(-nz)\right] \]

where \(m\) and \(n\) are integers; \(A, B, C, D\) are constants.
12 Functions of Several Variables

If \( \phi = f(x, y, z, \ldots) \) then \( \frac{\partial \phi}{\partial x} \) implies differentiation with respect to \( x \) keeping \( y, z, \ldots \) constant.

\[
\frac{\partial \phi}{\partial x} = \frac{\partial \phi}{\partial x} \frac{dx}{dx} + \frac{\partial \phi}{\partial y} \frac{dy}{dx} + \frac{\partial \phi}{\partial z} \frac{dz}{dx} + \cdots \text{ and } \frac{\delta \phi}{\delta x} = \frac{\partial \phi}{\partial x} \frac{dx}{dx} + \frac{\partial \phi}{\partial y} \frac{dy}{dx} + \frac{\partial \phi}{\partial z} \frac{dz}{dx} + \cdots
\]

where \( x, y, z, \ldots \) are independent variables. \( \frac{\partial \phi}{\partial x} \) is also written as \( \left( \frac{\partial \phi}{\partial x} \right)_{y, z, \ldots} \) or \( \frac{\partial \phi}{\partial x} \bigg|_{y, z, \ldots} \) when the variables kept constant need to be stated explicitly.

If \( \phi \) is a well-behaved function then \( \frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x} \) etc.

If \( \phi = f(x, y) \),

\[
\left( \frac{\partial \phi}{\partial x} \right)_{y} = \frac{1}{\left( \frac{\partial \phi}{\partial y} \right)_{x}}, \quad \left( \frac{\partial \phi}{\partial x} \right)_{y} \left( \frac{\partial \phi}{\partial y} \right)_{x} = -1.
\]

Taylor series for two variables

If \( \phi(x, y) \) is well-behaved in the vicinity of \( x = a, y = b \) then it has a Taylor series

\[
\phi(x, y) = \phi(a + u, b + v) = \phi(a, b) + u \frac{\partial \phi}{\partial x} + v \frac{\partial \phi}{\partial y} + \frac{1}{2!} \left( u^2 \frac{\partial^2 \phi}{\partial x^2} + 2uv \frac{\partial^2 \phi}{\partial x \partial y} + v^2 \frac{\partial^2 \phi}{\partial y^2} \right) + \cdots
\]

where \( x = a + u, y = b + v \) and the differential coefficients are evaluated at \( x = a, \quad y = b \)

Stationary points

A function \( \phi = f(x, y) \) has a stationary point when \( \frac{\partial \phi}{\partial x} = \frac{\partial \phi}{\partial y} = 0 \). Unless \( \frac{\partial^2 \phi}{\partial x^2} \neq \frac{\partial^2 \phi}{\partial y^2} \neq \frac{\partial^2 \phi}{\partial x \partial y} = 0 \), the following conditions determine whether it is a minimum, a maximum or a saddle point.

- Minimum: \( \frac{\partial^2 \phi}{\partial x^2} > 0, \quad \frac{\partial^2 \phi}{\partial y^2} > 0 \)
- Maximum: \( \frac{\partial^2 \phi}{\partial x^2} < 0, \quad \frac{\partial^2 \phi}{\partial y^2} < 0 \)
- Saddle point: \( \frac{\partial^2 \phi}{\partial x^2} \frac{\partial^2 \phi}{\partial y^2} < \left( \frac{\partial^2 \phi}{\partial x \partial y} \right)^2 \)

If \( \frac{\partial^2 \phi}{\partial x^2} = \frac{\partial^2 \phi}{\partial y^2} = \frac{\partial^2 \phi}{\partial x \partial y} = 0 \) the character of the turning point is determined by the next higher derivative.

Changing variables: the chain rule

If \( \phi = f(x, y, \ldots) \) and the variables \( x, y, \ldots \) are functions of independent variables \( u, v, \ldots \) then

\[
\frac{\partial \phi}{\partial u} = \frac{\partial \phi}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial \phi}{\partial y} \frac{\partial y}{\partial u} + \cdots
\]

\[
\frac{\partial \phi}{\partial v} = \frac{\partial \phi}{\partial x} \frac{\partial x}{\partial v} + \frac{\partial \phi}{\partial y} \frac{\partial y}{\partial v} + \cdots
\]

etc.
Changing variables in surface and volume integrals – Jacobians

If an area $A$ in the $x, y$ plane maps into an area $A'$ in the $u, v$ plane then

$$\int_A f(x, y) \, dx \, dy = \int_{A'} f(u, v) \, J \, du \, dv \quad \text{where} \quad J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

The Jacobian $J$ is also written as $\frac{\partial (x, y)}{\partial (u, v)}$. The corresponding formula for volume integrals is

$$\int_V f(x, y, z) \, dx \, dy \, dz = \int_{V'} f(u, v, w) \, J \, du \, dv \, dw \quad \text{where now} \quad J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$
13 Fourier Series and Transforms

Fourier series

If \( y(x) \) is a function defined in the range \( -\pi \leq x \leq \pi \) then

\[
y(x) \approx c_0 + \sum_{m=1}^{M} c_m \cos mx + \sum_{m=1}^{M'} s_m \sin mx
\]

where the coefficients are

\[
c_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} y(x) \, dx
\]

\[
c_m = \frac{1}{\pi} \int_{-\pi}^{\pi} y(x) \cos mx \, dx \quad (m = 1, \ldots, M)
\]

\[
s_m = \frac{1}{\pi} \int_{-\pi}^{\pi} y(x) \sin mx \, dx \quad (m = 1, \ldots, M')
\]

with convergence to \( y(x) \) as \( M, M' \to \infty \) for all points where \( y(x) \) is continuous.

Fourier series for other ranges

Variable \( t \), range \( 0 \leq t \leq T \), (i.e., a periodic function of time with period \( T \), frequency \( \omega = 2\pi/T \)).

\[
y(t) \approx c_0 + \sum c_m \cos \omega t + \sum s_m \sin \omega t
\]

where

\[
c_0 = \frac{\omega}{2\pi} \int_{0}^{T} y(t) \, dt, \quad c_m = \frac{\omega}{\pi} \int_{0}^{T} y(t) \cos \omega t \, dt, \quad s_m = \frac{\omega}{\pi} \int_{0}^{T} y(t) \sin \omega t \, dt.
\]

Variable \( x \), range \( 0 \leq x \leq L \),

\[
y(x) \approx c_0 + \sum c_m \cos \frac{2m\pi x}{L} + \sum s_m \sin \frac{2m\pi x}{L}
\]

where

\[
c_0 = \frac{1}{L} \int_{0}^{L} y(x) \, dx, \quad c_m = \frac{2}{L} \int_{0}^{L} y(x) \cos \frac{2m\pi x}{L} \, dx, \quad s_m = \frac{2}{L} \int_{0}^{L} y(x) \sin \frac{2m\pi x}{L} \, dx.
\]
Fourier series for odd and even functions

If \( y(x) \) is an odd (anti-symmetric) function [i.e., \( y(-x) = -y(x) \)] defined in the range \(-\pi \leq x \leq \pi\), then only sines are required in the Fourier series and \( s_m = \frac{2}{\pi} \int_{0}^{\pi} y(x) \sin mx \, dx \). If, in addition, \( y(x) \) is symmetric about \( x = \pi/2 \), then the coefficients \( s_m \) are given by \( s_m = 0 \) (for \( m \) even), \( s_m = \frac{4}{\pi} \int_{0}^{\pi/2} y(x) \sin mx \, dx \) (for \( m \) odd). If \( y(x) \) is an even (symmetric) function [i.e., \( y(-x) = y(x) \)] defined in the range \(-\pi \leq x \leq \pi\), then only constant and cosine terms are required in the Fourier series and \( c_0 = \frac{1}{\pi} \int_{0}^{\pi} y(x) \, dx \), \( c_m = \frac{2}{\pi} \int_{0}^{\pi} y(x) \cos mx \, dx \). If, in addition, \( y(x) \) is anti-symmetric about \( x = \pi/2 \), then \( c_0 = 0 \) and the coefficients \( c_m \) are given by \( c_m = 0 \) (for \( m \) even), \( c_m = \frac{4}{\pi} \int_{0}^{\pi/2} y(x) \cos mx \, dx \) (for \( m \) odd).

[These results also apply to Fourier series with more general ranges provided appropriate changes are made to the limits of integration.]

Complex form of Fourier series

If \( y(x) \) is a function defined in the range \(-\pi \leq x \leq \pi\) then
\[
y(x) \approx \sum_{m=-\infty}^{M} C_m e^{imx}, \quad C_m = \frac{1}{2\pi} \int_{-\pi}^{\pi} y(x) e^{-imx} \, dx
\]
with \( m \) taking all integer values in the range \( \pm M \). This approximation converges to \( y(x) \) as \( M \to \infty \) under the same conditions as the real form.

For other ranges the formulae are:

Variable \( t \), range \( 0 \leq t \leq T \), frequency \( \omega = 2\pi/T \),
\[
y(t) = \sum_{m=-\infty}^{\infty} C_m e^{im\omega t}, \quad C_m = \frac{\omega}{2\pi} \int_{0}^{T} y(t) e^{-im\omega t} \, dt.
\]

Variable \( x' \), range \( 0 \leq x' \leq L \),
\[
y(x') = \sum_{m=-\infty}^{\infty} C_m e^{im2\pi x' / L}, \quad C_m = \frac{1}{L} \int_{0}^{L} y(x') e^{-im2\pi x' / L} \, dx'.
\]

Discrete Fourier series

If \( y(x) \) is a function defined in the range \(-\pi \leq x \leq \pi\) which is sampled in the \( 2N \) equally spaced points \( x_n = nx/N \quad [n = -(N - 1), \ldots, N] \), then
\[
y(x_n) = c_0 + c_1 \cos x_n + c_2 \cos 2x_n + \cdots + c_{N-1} \cos (N-1)x_n + c_N \cos Nx_n
\]
\[
+ s_1 \sin x_n + s_2 \sin 2x_n + \cdots + s_{N-1} \sin (N-1)x_n + s_N \sin Nx_n
\]
where the coefficients are
\[
c_0 = \frac{1}{2N} \sum_{n} y(x_n)
\]
\[
c_m = \frac{1}{N} \sum_{n} y(x_n) \cos mx_n \quad (m = 1, \ldots, N - 1)
\]
\[
c_N = \frac{1}{2N} \sum_{n} y(x_n) \cos Nx_n
\]
\[
s_m = \frac{1}{N} \sum_{n} y(x_n) \sin mx_n \quad (m = 1, \ldots, N - 1)
\]
\[
s_N = \frac{1}{2N} \sum_{n} y(x_n) \sin Nx_n
\]
each summation being over the \( 2N \) sampling points \( x_n \).
Fourier transforms

If \( y(x) \) is a function defined in the range \(-\infty \leq x \leq \infty\) then the Fourier transform \( \hat{y}(\omega) \) is defined by the equations

\[
y(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{y}(\omega) e^{i\omega t} \, d\omega, \quad \hat{y}(\omega) = \int_{-\infty}^{\infty} y(t) e^{-i\omega t} \, dt.
\]

If \( \omega \) is replaced by \( 2\pi f \), where \( f \) is the frequency, this relationship becomes

\[
y(t) = \int_{-\infty}^{\infty} \hat{y}(f) e^{i2\pi ft} \, df, \quad \hat{y}(f) = \int_{-\infty}^{\infty} y(t) e^{-i2\pi ft} \, dt.
\]

If \( y(t) \) is symmetric about \( t = 0 \) then

\[
y(t) = \frac{1}{\pi} \int_{0}^{\infty} \hat{y}(\omega) \cos \omega t \, d\omega, \quad \hat{y}(\omega) = 2 \int_{0}^{\infty} y(t) \cos \omega t \, dt.
\]

If \( y(t) \) is anti-symmetric about \( t = 0 \) then

\[
y(t) = \frac{1}{\pi} \int_{0}^{\infty} \hat{y}(\omega) \sin \omega t \, d\omega, \quad \hat{y}(\omega) = 2 \int_{0}^{\infty} y(t) \sin \omega t \, dt.
\]

Specific cases

\[y(t) = a, \quad |t| \leq \tau, \quad (\text{Top Hat}),\]
\[\hat{y}(\omega) = 2a \frac{\sin \omega \tau}{\omega}, \quad \hat{y}(\omega) = 2a \frac{\sin \omega \tau}{\omega} \equiv 2a \tau \text{sinc}(\omega \tau)\]

where \( \text{sinc}(x) = \frac{\sin(x)}{x} \)

\[y(t) = a(1 - |t|/\tau), \quad |t| \leq \tau, \quad (\text{Saw-tooth}),\]
\[\hat{y}(\omega) = \frac{2a}{\omega^2 \tau} (1 - \cos \omega \tau) = a \tau \text{sinc}^2 \left( \frac{\omega \tau}{2} \right)\]

\[y(t) = \exp(-t^2/\tau^2) \quad (\text{Gaussian}),\]
\[\hat{y}(\omega) = t_0 \sqrt{\pi} \exp \left(-\omega^2 \tau^2/4\right)\]

\[y(t) = f(t) e^{i\omega t} \quad (\text{modulated function}),\]
\[\hat{y}(\omega) = \hat{f}(\omega - \omega_0)\]

\[y(t) = \sum_{m=-\infty}^{\infty} \delta(t - m\tau) \quad (\text{sampling function}),\]
\[\hat{y}(\omega) = \sum_{n=-\infty}^{\infty} \delta(\omega - 2\pi n/\tau)\]
Convolution theorem

If \( z(t) = \int_{-\infty}^{\infty} x(\tau)y(t-\tau) \, d\tau = \int_{-\infty}^{\infty} x(t-\tau)y(\tau) \, d\tau \equiv x(t) \ast y(t) \) then \( \hat{z}(\omega) = \hat{x}(\omega) \hat{y}(\omega) \).

Conversely, \( \hat{x \ast y} = \hat{x} \ast \hat{y} \).
14 Laplace Transforms

If \( y(t) \) is a function defined for \( t \geq 0 \), the Laplace transform \( \mathcal{L}\{y(t)\} = Y(s) \) is defined by the equation

\[
Y(s) = \mathcal{L}\{y(t)\} = \int_0^\infty e^{-st}y(t)\,dt
\]

<table>
<thead>
<tr>
<th>Function ( y(t) ) ( \ (t &gt; 0) )</th>
<th>Transform ( Y(s) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \delta(t) ) ( \ (t &gt; 0) )</td>
<td>1</td>
</tr>
<tr>
<td>( \theta(t) ) ( \ (t &gt; 0) )</td>
<td>( \frac{1}{s} )</td>
</tr>
<tr>
<td>( t^n ) ( \ (t &gt; 0) )</td>
<td>( \frac{n!}{s^{n+1}} )</td>
</tr>
<tr>
<td>( t^{\frac{1}{2}} ) ( \ (t &gt; 0) )</td>
<td>( \frac{1}{2} \sqrt{\frac{\pi}{s^3}} )</td>
</tr>
<tr>
<td>( t^{-\frac{1}{2}} ) ( \ (t &gt; 0) )</td>
<td>( \sqrt{\frac{\pi}{s}} )</td>
</tr>
<tr>
<td>( e^{-at} ) ( \ (t &gt; 0) )</td>
<td>( \frac{1}{s+a} )</td>
</tr>
<tr>
<td>( \sin \omega t ) ( \ (t &gt; 0) )</td>
<td>( \frac{\omega}{s^2 + \omega^2} )</td>
</tr>
<tr>
<td>( \cos \omega t ) ( \ (t &gt; 0) )</td>
<td>( \frac{s}{s^2 + \omega^2} )</td>
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<td>( \frac{\omega}{s^2 - \omega^2} )</td>
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<td>( \frac{s}{s^2 - \omega^2} )</td>
</tr>
<tr>
<td>( e^{-at}y(t) ) ( \ (t &gt; 0) )</td>
<td>( e^{-at}Y(s) )</td>
</tr>
<tr>
<td>( y(t-\tau)\theta(t-\tau) ) ( \ (t &gt; 0) )</td>
<td>( e^{-\tau s} \mathcal{L}{y(t)} )</td>
</tr>
<tr>
<td>( ty(t) ) ( \ (t &gt; 0) )</td>
<td>( -\frac{dY}{ds} )</td>
</tr>
<tr>
<td>( \frac{dy}{dt} ) ( \ (t &gt; 0) )</td>
<td>( sY(s) - y(0) )</td>
</tr>
<tr>
<td>( \frac{d^n y}{dt^n} ) ( \ (t &gt; 0) )</td>
<td>( s^nY(s) - s^{n-1}y(0) - s^{n-2}\left[ \frac{dy}{dt} \right]_0 - \cdots - \left[ \frac{d^{n-1} y}{dt^{n-1}} \right]_0 )</td>
</tr>
<tr>
<td>( \int_0^t y(\tau),d\tau ) ( \ (t &gt; 0) )</td>
<td>( \frac{Y(s)}{s} )</td>
</tr>
<tr>
<td>( \int_0^t x(\tau),y(t-\tau),d\tau ) ( \ (t &gt; 0) )</td>
<td>( \bar{X}(s)Y(s) ) Convolution theorem</td>
</tr>
<tr>
<td>( \int_0^t x(t-\tau),y(\tau),d\tau ) ( \ (t &gt; 0) )</td>
<td>( \bar{X}(s) \mathcal{L}{y(t)} )</td>
</tr>
</tbody>
</table>

[Note that if \( y(t) = 0 \) for \( t < 0 \) then the Fourier transform of \( y(t) \) is \( \hat{y}(\omega) = Y(i\omega) \).]
15 Numerical Analysis

Finding the zeros of equations

If the equation is \( y = f(x) \) and \( x_n \) is an approximation to the root then either

\[
x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.
\]

(Newton)

or,

\[
x_{n+1} = x_n - \frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})} f(x_n)
\]

(Linear interpolation)

are, in general, better approximations.

Numerical integration of differential equations

If \( \frac{dy}{dx} = f(x, y) \) then

\[
y_{n+1} = y_n + hf(x_n, y_n)
\]

where \( h = x_{n+1} - x_n \)

(Euler method)

Putting \( y'_{n+1} = y_n + hf(x_n, y_n) \)

then

\[
y_{n+1} = y_n + \frac{h[f(x_n, y_n) + f(x_{n+1}, y'_{n+1})]}{2}
\]

(improved Euler method)

Numerical evaluation of definite integrals

Trapezoidal rule

The interval of integration is divided into \( n \) equal sub-intervals, each of width \( h \); then

\[
\int_a^b f(x) \, dx \approx h \left[ \frac{1}{2} f(a) + f(x_1) + \cdots + f(x_i) + \cdots + \frac{1}{2} f(b) \right]
\]

where \( h = (b - a)/n \) and \( x_j = a + jh \).

Simpson's rule

The interval of integration is divided into an even number (say \( 2n \)) of equal sub-intervals, each of width \( h = (b - a)/2n \); then

\[
\int_a^b f(x) \, dx \approx \frac{h}{3} \left[ f(a) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \cdots + 2f(x_{2n-2}) + 4f(x_{2n-1}) + f(b) \right]
\]

Gauss's integration formulae

These have the general form

\[
\int_{-1}^{1} y(x) \, dx \approx \sum_{i=1}^{n} c_i y(x_i)
\]

For \( n = 2 \):

\( x_i = \pm 0.5773, \quad c_i = 1, 1 \) (exact for any cubic).

For \( n = 3 \):

\( x_i = -0.7746, 0, 0.7746; \quad c_i = 0.555, 0.888, 0.555 \) (exact for any quintic).